

# 9

## Properties of the Circle

### TERMINOLOGY

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**Arc:** Part of a curve, most commonly a portion of the distance around the circumference of a circle

**Chord:** A straight line joining two points on the circumference of a circle

**Concentric circles:** Circles that have the same centre

**Concyclic points:** Points that lie on the circumference of the same circle

**Cyclic quadrilateral:** A cyclic quadrilateral is a figure whose four vertices are concyclic points. The four vertices lie on the circumference of a circle

**Radius:** A radius is the distance from the centre of a circle out to the circumference (radii is plural, meaning more than one radius)

**Subtend:** Form an angle at some point (usually the centre or circumference of a circle)

**Tangent:** A straight line external to a curve or circle that just touches the curve or circle at a single point

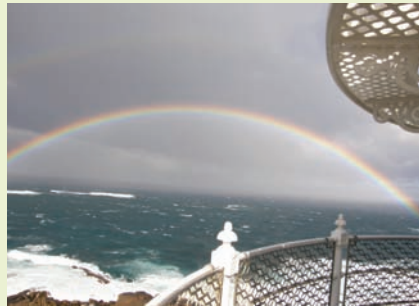


## INTRODUCTION

IN CHAPTER 4, YOU STUDIED the geometry of angles, triangles, quadrilaterals and other polygons. This chapter shows you some properties of the circle.

### DID YOU KNOW?

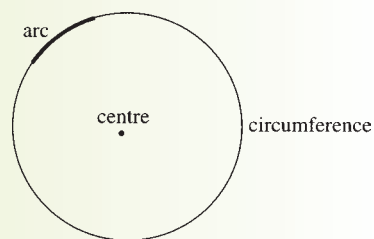
A rainbow is the shape of an arc of a circle. If you could see the whole rainbow, it would form a circle. Research the rainbow on the Internet and find out more about its shape and other properties.



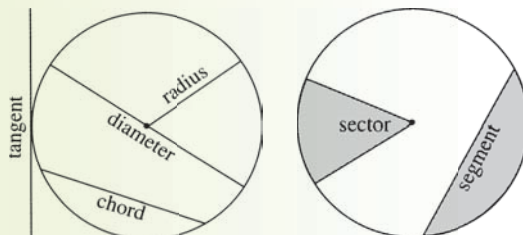
### Parts of a Circle

EXTENSION

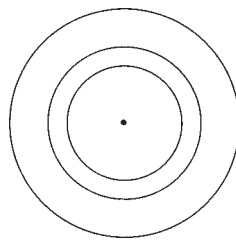
An **arc** is a part of the **circumference**.



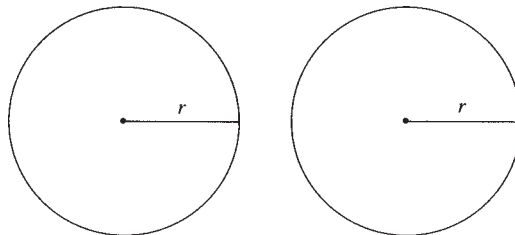
A **tangent** touches the circle at **one point**.



**Concentric circles** are circles that have the same **centre**.



**Equal circles** have the same **radius**.



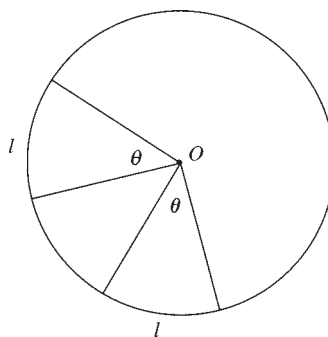
### DID YOU KNOW?

**Leonardo da Vinci** (1452–1519) was a great artist, scientist and inventor. He studied geometry, and many of his model drawings show this influence. His drawings included designs for flying machines, spring-driven automobiles, bridges and weapons. Leonardo's designs were revolutionary, and the scientists of his time did not have the knowledge needed to make the models work.

## Arcs, Angles and Chords

EXTENSION

Equal arcs subtend equal angles at the centre of the circle.



**Proof**

Let two equal arcs have lengths  $l_1$  and  $l_2$ , and subtend angles of  $\alpha$  and  $\beta$  at the centre of the circle.

Using  $l = r\theta$ ,  $l_1 = r\alpha$  and  $l_2 = r\beta$

But  $l_1 = l_2$

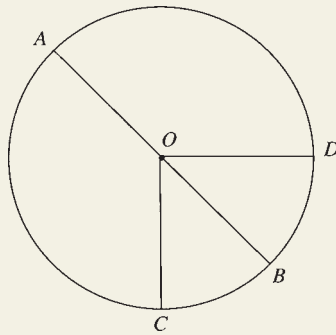
$\therefore r\alpha = r\beta$

$\therefore \alpha = \beta$

The converse is also true:

You will study the formula  
 $l = r\theta$  in Chapter 5 of  
the HSC Course book.

If two arcs subtend equal angles at the centre of the circle, then the arcs are equal.

**EXAMPLE**

$AB$  is a diameter of the circle with centre  $O$ . Arc  $CB = \text{arc } BD$ .  
Prove  $\angle AOC = \angle AOD$ .

**Solution**

Since arc  $CB = \text{arc } BD$ ,  $\angle COB = \angle DOB$

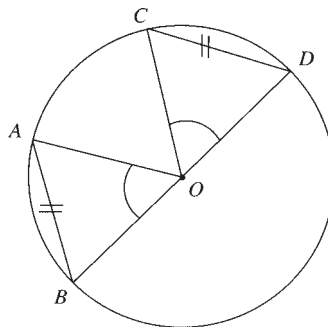
Let  $\angle COB = \angle DOB = x$

Then  $\angle AOC = 180^\circ - \angle COB$  ( $\angle AOB$  is a straight angle)  
 $= 180^\circ - x$

Also  $\angle AOD = 180^\circ - \angle DOB$  (similarly)  
 $= 180^\circ - x$

$\therefore \angle AOC = \angle AOD$

Equal chords subtend equal angles at the centre of the circle.



### Proof

$$OA = OC \quad \text{(equal radii)}$$

$$OB = OD \quad \text{(similarly)}$$

$$AB = CD \quad \text{(given)}$$

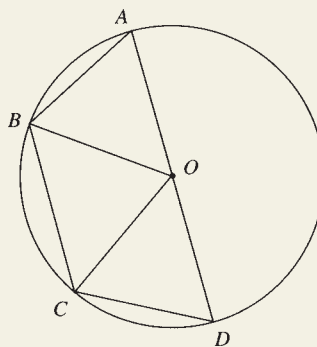
$$\therefore \text{ by SSS, } \triangle OAB \cong \triangle OCD$$

$$\therefore \angle AOB = \angle COD \quad \text{(corresponding } \angle \text{s in congruent } \triangle \text{s)}$$

The converse is also true:

Equal angles subtended at the centre of the circle cut off equal chords.

### EXAMPLE



$AD$  is a diameter of the circle with centre  $O$ , where  $AB = CD$ . Prove that  $BC \parallel AD$ .

**Solution**

Since  $AB = CD$ ,  $\angle AOB = \angle COD$

Let  $\angle AOB = \angle COD = x$

Then  $\angle BOC = 180^\circ - (x + x)$  ( $\angle AOD$  is a straight  $\angle$ )

$$= 180^\circ - 2x$$

$OB = OC$  (equal radii)

$\therefore \triangle OBC$  is isosceles with  $\angle OBC = \angle OCB$

$\angle OBC + \angle OCB + 180^\circ - 2x = 180^\circ$  ( $\angle$  sum of  $\triangle OBC$ )

$$\angle OBC + \angle OCB = 2x$$

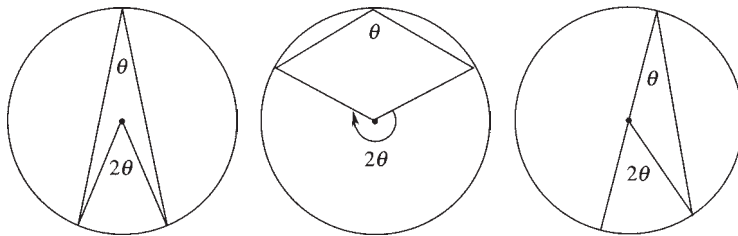
$\therefore \angle OBC = \angle OCB = x$

$\therefore \angle OBC = \angle AOB$

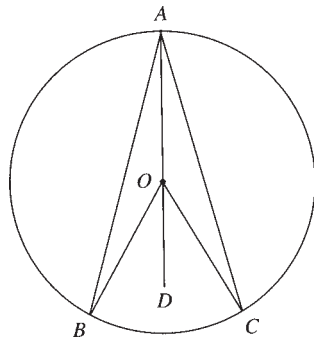
But these are equal alternate angles

$\therefore BC \parallel AD$

The angle at the centre of a circle is twice the angle at the circumference subtended by the same arc.



*These figures show that this property can look quite different in different situations.*

**Proof**

Join  $AO$  and produce to  $D$ .

Let  $\angle BAO = x$  and  $\angle CAO = y$

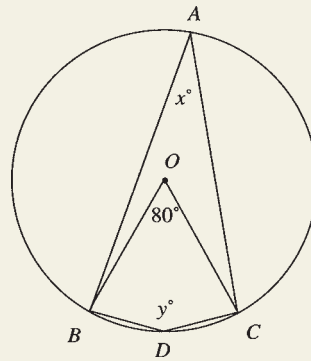
$\therefore \angle BAC = x + y$

$OA = OB$  (equal radii)

$$\begin{aligned}
 \therefore \angle OBA &= x && \text{(base } \angle \text{ s of isosceles } \Delta) \\
 &= \angle BAO \\
 OA &= OC && \text{(equal radii)} \\
 \therefore \angle OCA &= y && \text{(base } \angle \text{ s of isosceles } \Delta) \\
 &= \angle CAO \\
 \angle BOD &= x + x && \text{(exterior } \angle \text{ of } \Delta OBA) \\
 &= 2x \\
 \angle COD &= y + y && \text{(exterior } \angle \text{ of } \Delta OCA) \\
 &= 2y \\
 \angle BOC &= \angle BOD + \angle COD \\
 &= 2x + 2y \\
 &= 2(x + y) \\
 &= 2\angle BAC
 \end{aligned}$$

## EXAMPLES

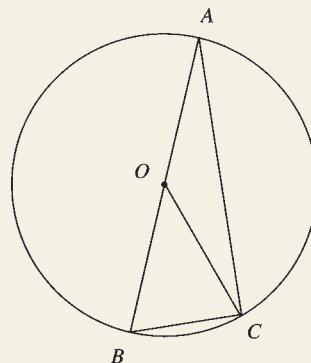
1. Find the values of  $x$  and  $y$ .



### Solution

$$\begin{aligned}
 x &= 40 && \text{(\angle at centre is twice the } \angle \text{ at the circumference)} \\
 \text{Reflex } \angle BOC &= 360^\circ - 80^\circ && \text{(\angle of revolution)} \\
 &= 280^\circ \\
 \therefore y &= 140 && \text{(\angle at centre is twice the } \angle \text{ at the circumference)}
 \end{aligned}$$

2. Prove  $\angle BOC$  is twice the size of  $\angle OCA$ .



**Solution**

Let  $\angle OAC = x$

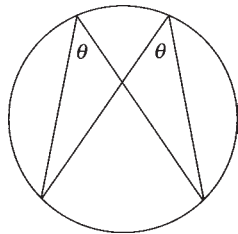
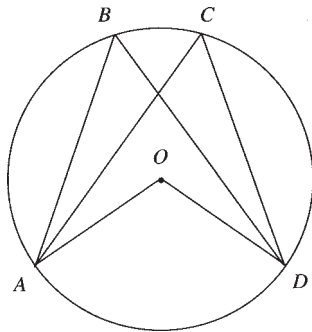
$$\begin{aligned}\angle BOC &= 2\angle OAC && (\angle \text{ at centre is twice the } \angle \text{ at the circumference}) \\ &= 2x\end{aligned}$$

$$OA = OC \quad (\text{equal radii})$$

$\therefore \angle OCA = \angle OAC = x$  (base  $\angle$ s of isosceles  $\Delta$ )

$\therefore \angle BOC = 2\angle OCA$

Angles in the same segment of a circle are equal.

**Proof**

Join  $A$  and  $D$  to centre  $O$

$$\angle AOD = 2\angle ABD \quad (\angle \text{ at centre is twice the } \angle \text{ at the circumference})$$

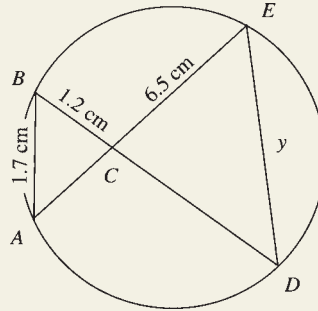
$$\angle AOD = 2\angle ACD$$

$\therefore \angle ABD = \angle ACD$



## EXAMPLE

Prove  $\triangle ABC$  and  $\triangle DEC$  are similar. Hence find the value of  $y$  correct to 1 decimal place.



## Solution

$$\angle ABC = \angle DEC \quad (\angle \text{s in same segment})$$

$$\angle BCA = \angle ECD \quad (\text{vertically opposite } \angle \text{s})$$

$$\therefore \triangle ABC \parallel \triangle DEC$$

$$\therefore \frac{ED}{BA} = \frac{EC}{BC}$$

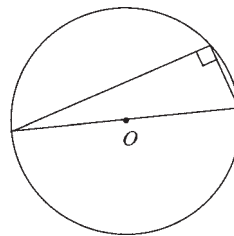
$$\frac{y}{1.7} = \frac{6.5}{1.2}$$

$$1.2y = 1.7 \times 6.5$$

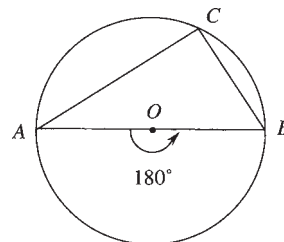
$$y = \frac{1.7 \times 6.5}{1.2}$$

$$= 9.2 \text{ cm correct to 1 decimal place.}$$

The angle in a semicircle is a right angle.



## Proof



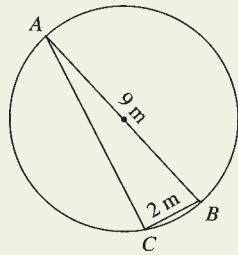
$$\angle AOB = 180^\circ \quad (\text{straight } \angle)$$

$$\angle AOB = 2\angle ACB \quad (\angle \text{ at centre is twice the } \angle \text{ at the circumference})$$

$$\therefore \angle ACB = 90^\circ$$

### EXAMPLE

$AB$  is a diameter of the circle below. If  $BC = 2$  m and  $AB = 9$  m, find the exact length of  $AC$ .



### Solution

$$\angle ACB = 90^\circ$$

$$\therefore AB^2 = AC^2 + BC^2$$

$$9^2 = AC^2 + 2^2$$

$$81 = AC^2 + 4$$

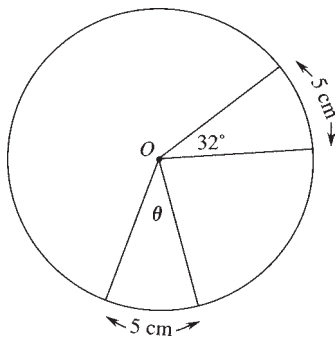
$$77 = AC^2$$

$$\therefore AC = \sqrt{77} \text{ m}$$

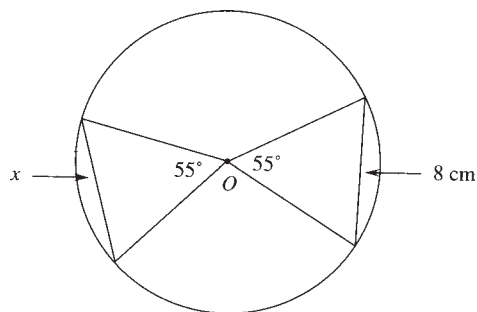
## 9.1 Exercises

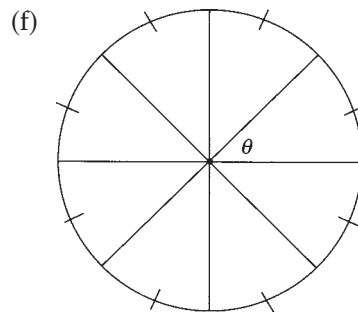
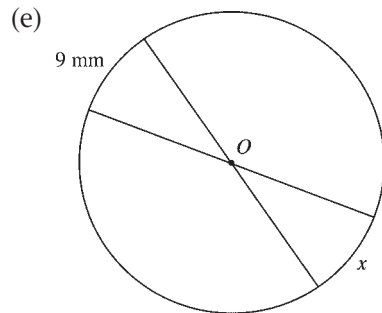
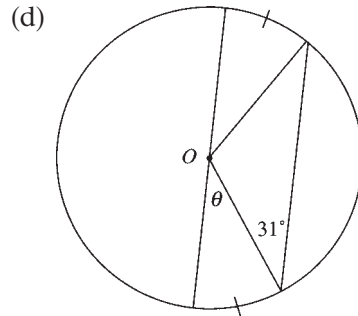
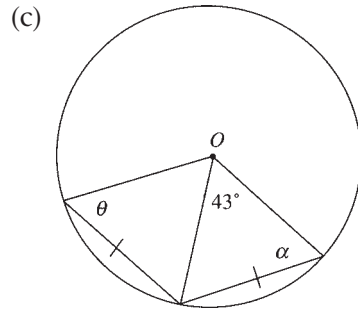
1. Find values of all pronumerals ( $O$  is the centre of each circle).

(a)



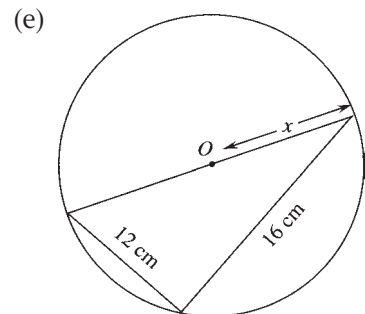
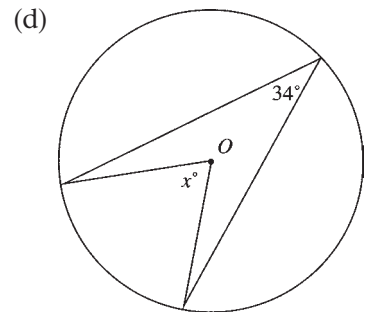
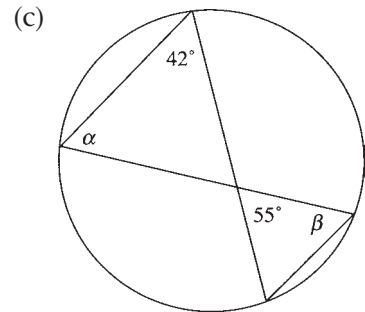
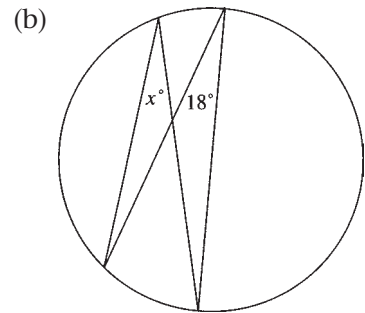
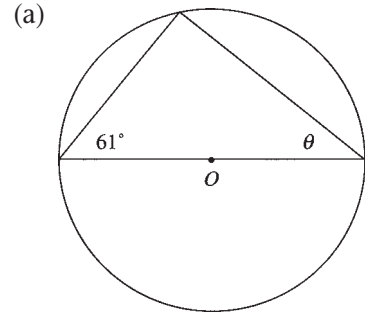
(b)

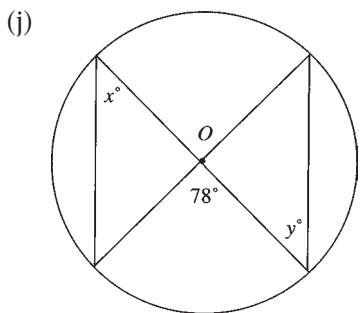
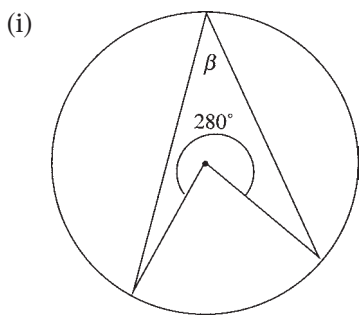
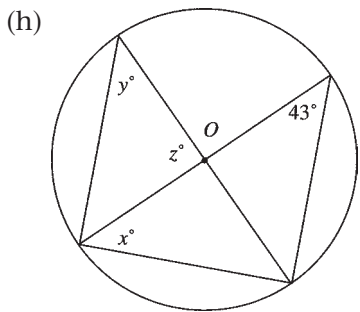
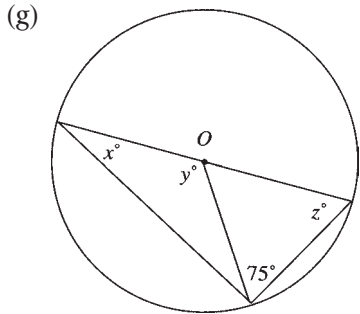
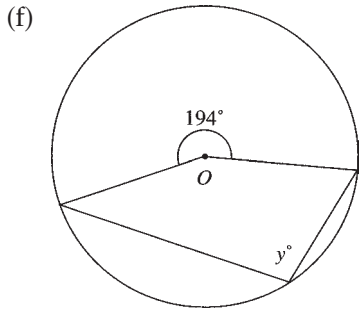




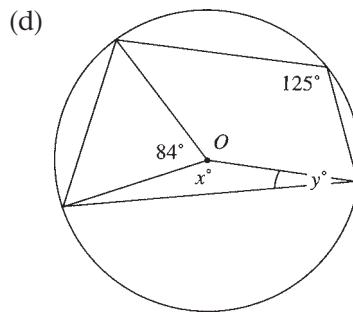
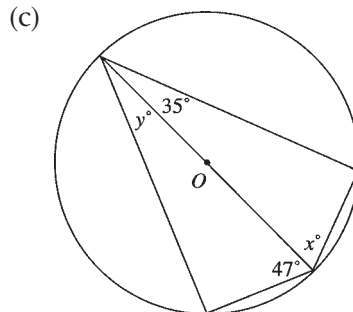
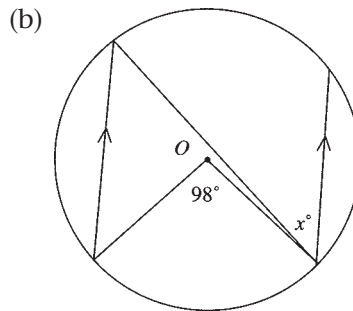
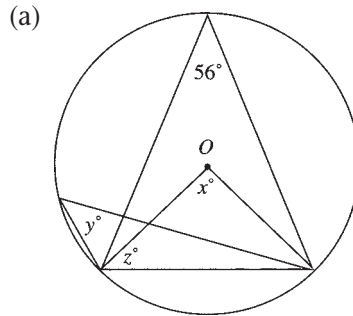
2. The circumference of a circle is  $16\pi$  cm. Find the length of the arc that subtends an angle of  $40^\circ$  at the centre of the circle.

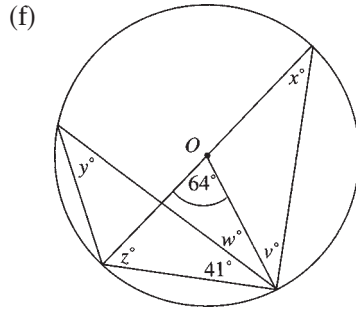
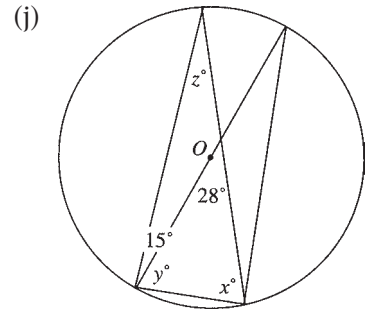
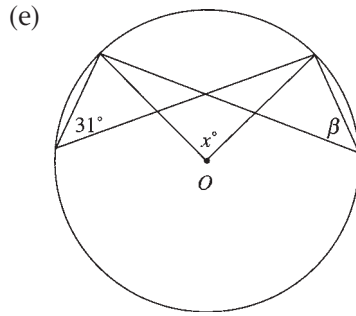
3. Find values of all pronumerals (O is the centre of each circle).



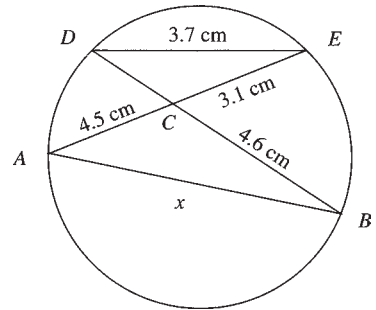


4. Find values of all pronumerals (O is the centre of each circle).

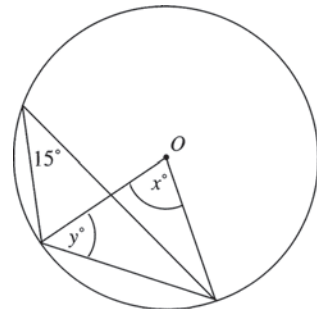
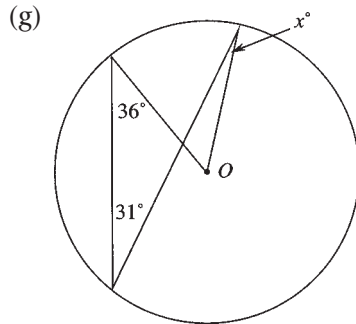




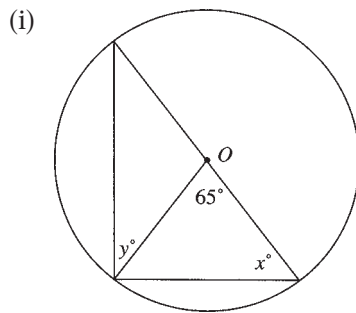
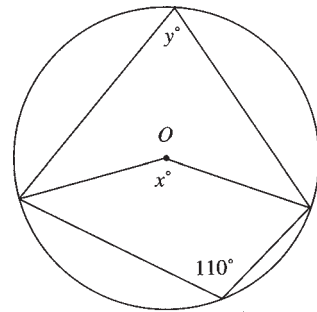
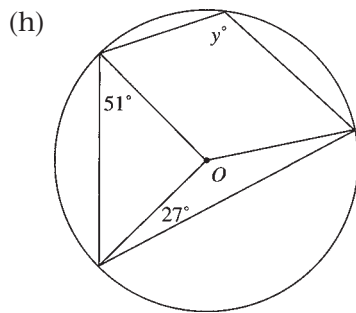
5. (a) Prove  $\triangle ABC \parallel \triangle DEC$ .  
 (b) Hence find the value of  $x$  correct to 1 decimal place.



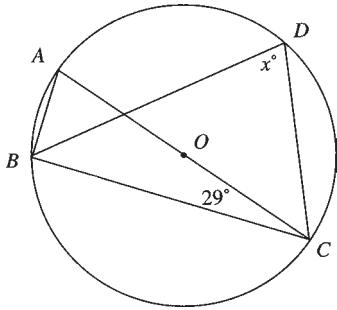
6. Find  $x$  and  $y$ , giving reasons.



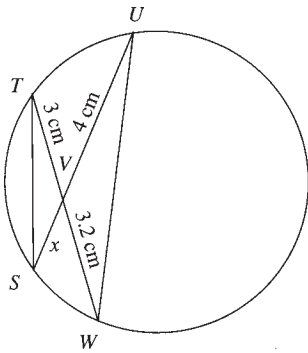
7. Find  $x$  and  $y$ , giving reasons.



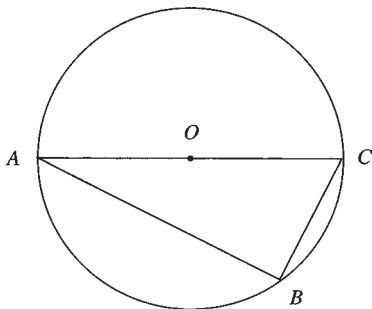
8. Evaluate  $x$ , giving reasons for each step in your calculation.



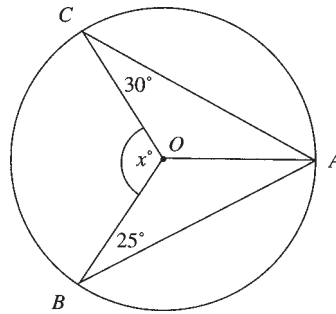
9. Prove  $\Delta STV$  and  $\Delta WUV$  are similar. Hence find  $x$ .



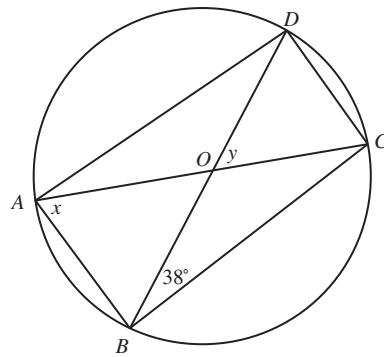
10.  $AB = 6$  cm and  $BC = 3$  cm.  $O$  is the centre of the circle. Show that the radius of the circle is  $\frac{3\sqrt{5}}{2}$  cm.



11. Find  $x$ , giving reasons for each step in your calculations.

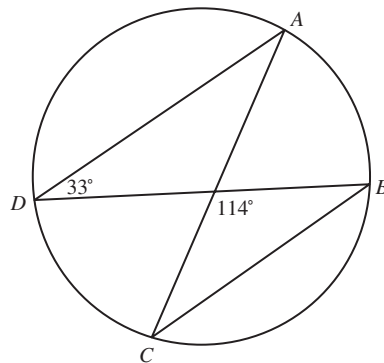


12. The circle below has centre  $O$ .

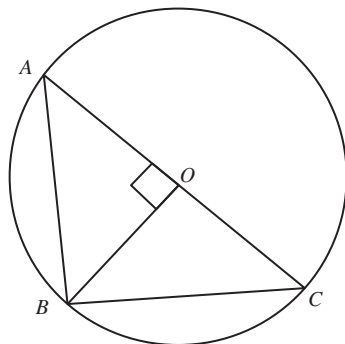


- (a) Evaluate  $x$  and  $y$ .  
 (b) Show that  $AD = BC$ .

13. Show that  $AD \parallel BC$  in the circle below.

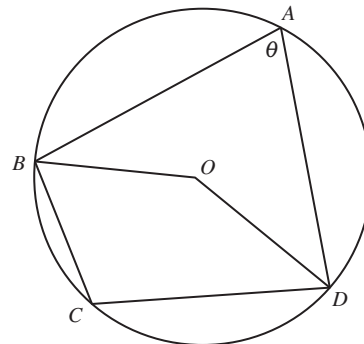


14. A circle has centre  $O$  and radius  $r$  as shown.



- (a) Show that triangles  $AOB$  and  $ABC$  are similar.  
 (b) Show that  $BC = \sqrt{2}r$ .

15. The circle below has centre  $O$  and  $\angle DAB = \theta$ .

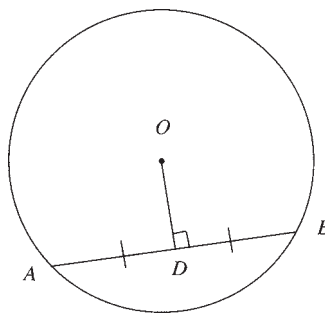


Show that  $\angle DAB$  and  $\angle BCD$  are supplementary.

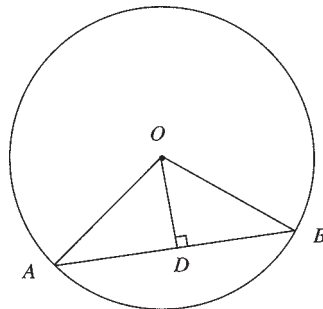
## Chord Properties

EXTENSION

A perpendicular line from the centre of a circle to a chord bisects the chord.



*Proof*



$$\begin{aligned}\angle ADO &= \angle BDO = 90^\circ && \text{(given)} \\ OA &= OB && \text{(equal radii)}\end{aligned}$$

$OD$  is common

$\therefore$  by RHS  $\triangle OAD \equiv \triangle OBD$

$\therefore AD = BD$  (corresponding sides in congruent  $\triangle$ s)

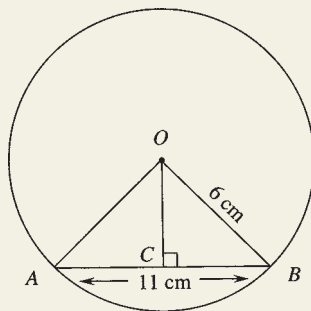
So  $OD$  bisects  $AB$

The converse is also true:

A line from the centre of a circle that bisects a chord is perpendicular to the chord.

## EXAMPLES

1. Line  $OC$  is perpendicular to chord  $AB$ . If the radius of the circle is 6 cm and the chord is 11 cm long, find the length of  $OC$ , correct to 1 decimal place.



### Solution

$$AB = 11$$

$$\therefore AC = 5.5 \quad (\text{OC bisects AB})$$

$$\text{Also } OA = 6 \quad (\text{radius—given})$$

$$OA^2 = AC^2 + OC^2$$

$$6^2 = 5.5^2 + OC^2$$

$$36 = 30.25 + OC^2$$

$$5.75 = OC^2$$

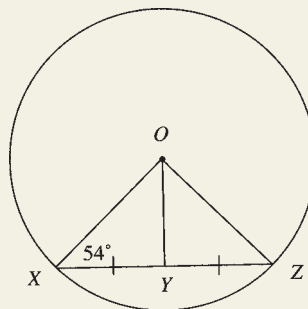
$$\begin{aligned}\therefore OC &= \sqrt{5.75} \\ &= 2.4 \text{ cm}\end{aligned}$$

CONTINUED





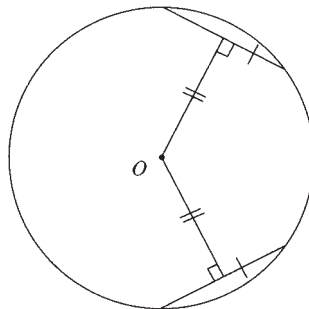
2. Given  $XY = YZ$  and  $\angle OXY = 54^\circ$ , find  $\angle XOY$ .



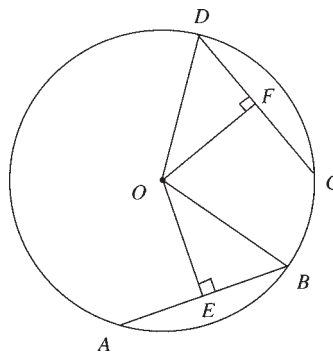
### Solution

$$\begin{aligned} \angle OYX &= 90^\circ && \text{(OY bisects XZ)} \\ \angle XOY &= 180^\circ - (90^\circ + 54^\circ) && \text{(\angle sum of } \triangle OXY) \\ &= 36^\circ \end{aligned}$$

Equal chords are equidistant from the centre of the circle.



### Proof



$$\begin{aligned} \text{Let } CD &= AB \\ \angle OEB &= \angle OFD = 90^\circ && \text{(given)} \\ OB &= OD && \text{(equal radii)} \\ AB &= CD && \text{(given)} \end{aligned}$$

$$BE = \frac{1}{2}AB \quad (OE \text{ bisects } AB)$$

$$DF = \frac{1}{2}CD \quad (OF \text{ bisects } CD)$$

$$\therefore BE = DF$$

$$\therefore \text{by RHS } \triangle OEB \equiv \triangle OFD$$

$$\therefore OE = OF \quad (\text{corresponding sides in congruent } \triangle s)$$

The converse is also true:

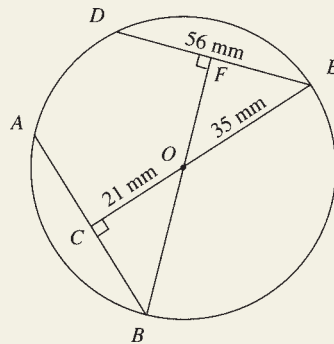
Chords that are equidistant from the centre are equal.

### Class Exercise

Prove that chords that are equidistant from the centre are equal.

### EXAMPLE

In the circle below, with centre  $O$ ,  $OE = 35$  mm,  $DE = 56$  mm and  $OC = 21$  mm. Show that  $AB = DE$ .



### Solution

$$EF = 28 \text{ mm} \quad (OF \text{ bisects } DE)$$

$$OE^2 = EF^2 + OF^2$$

$$35^2 = 28^2 + OF^2$$

$$35^2 - 28^2 = OF^2$$

$$441 = OF^2$$

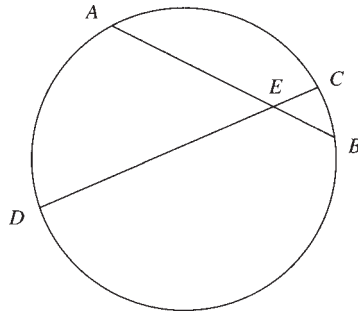
$$\therefore OF = \sqrt{441}$$

$$= 21 \text{ mm}$$

$$\therefore OF = OC \quad (\text{chords equal when equidistant from the centre})$$

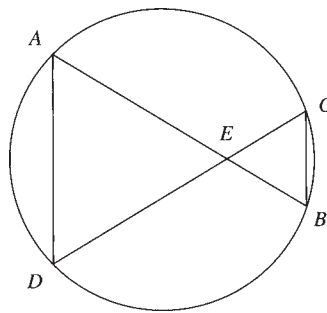
$$\text{So } AB = DE.$$

The products of intercepts of intersecting chords are equal.



$$AE \cdot EB = DE \cdot EC$$

**Proof**



$$\angle AED = \angle CEB$$

(vertically opposite  $\angle$ s)

$$\angle DAE = \angle ECB$$

( $\angle$ s in same segment)

$$\angle ADE = \angle EBC$$

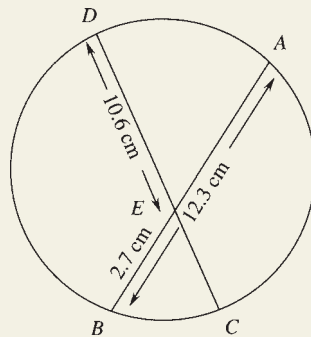
(similarly)

$$\therefore \triangle AED \parallel \triangle CEB$$

$$\therefore \frac{AE}{EC} = \frac{DE}{EB}$$

$$\therefore AE \cdot EB = DE \cdot EC$$

### EXAMPLE



Given chord  $AB = 12.3$  cm,  $EB = 2.7$  cm and  $DE = 10.6$  cm, find the length of  $EC$ , correct to 1 decimal place.

**Solution**

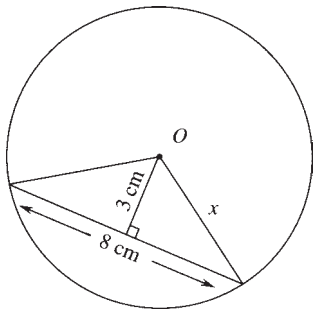
$$\begin{aligned} AE &= AB - EB \\ &= 12.3 - 2.7 \\ &= 9.6 \end{aligned}$$

$$\begin{aligned} AE \cdot EB &= DE \cdot EC \\ 9.6 \times 2.7 &= 10.6 \times EC \\ \therefore EC &= \frac{9.6 \times 2.7}{10.6} \\ &= 2.4 \text{ cm} \end{aligned}$$

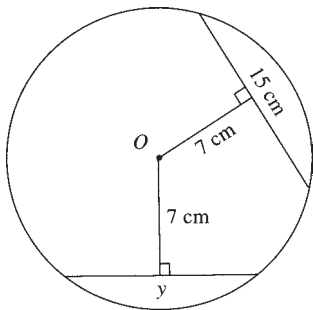
**9.2 Exercises**

1. Find the values of all pronumerals ( $O$  is the centre of each circle).

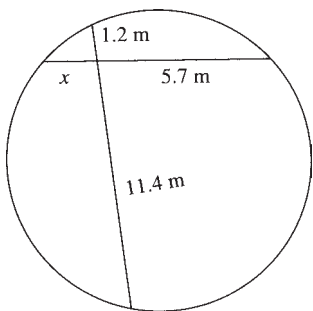
(a)



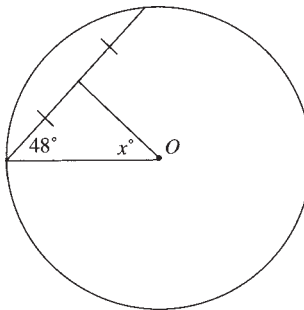
(b)



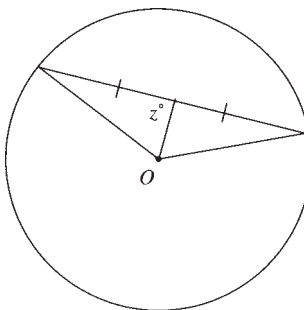
(c)



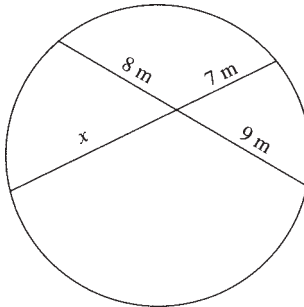
(d)

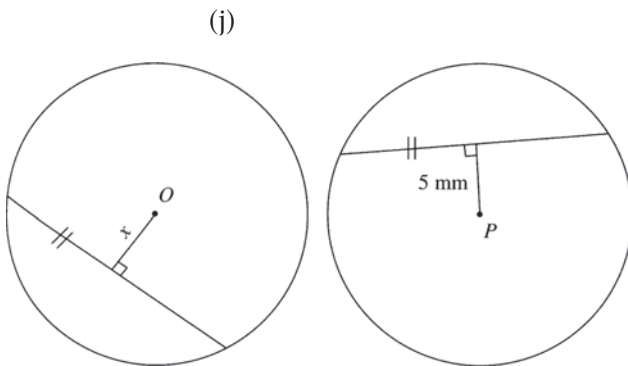
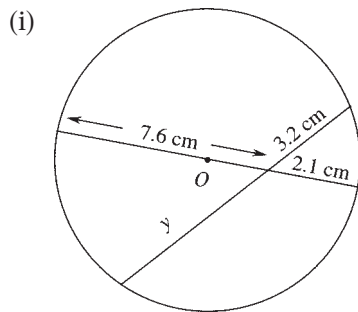
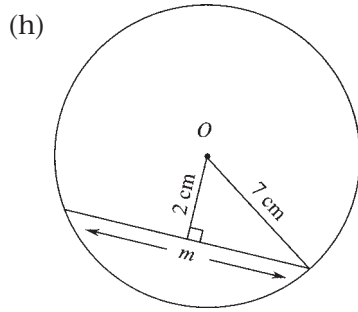
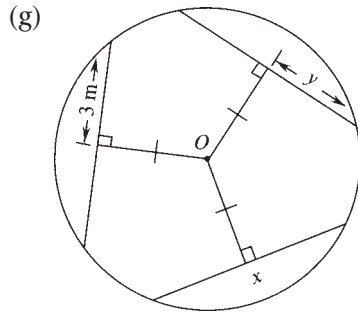


(e)



(f)

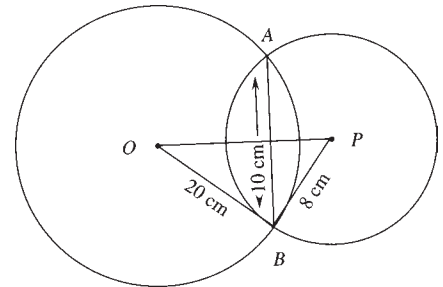




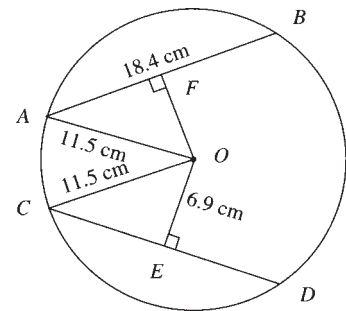
2. Find the exact radius of a circle with a chord that is 8 cm long and 5 cm from the centre.

3. A circle with radius 89 mm has a chord drawn 52 mm from the centre. How long, to the nearest millimetre, is the chord?

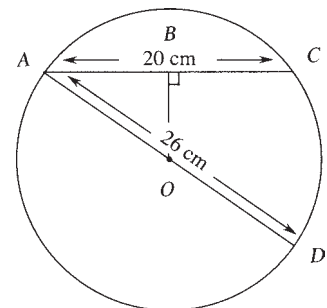
4.  $O$  and  $P$  are the centres of intersecting circles with radii 20 cm and 8 cm respectively. If  $AB = 10$  cm, find the distance  $OP$ , correct to 1 decimal place.



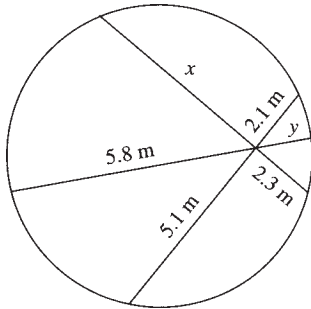
5. Show  $AB = CD$ .



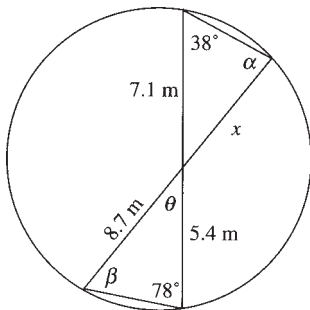
6.  $AC = 20$  cm and  $AD = 26$  cm. Find  $OB$ , correct to 1 decimal place.



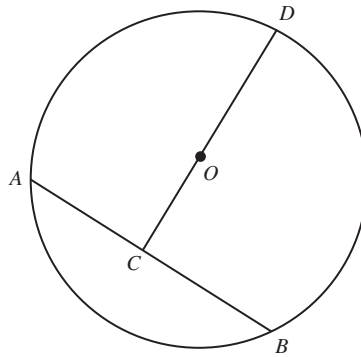
7. Evaluate  $x$  and  $y$ , correct to 1 decimal place.



8. Find the values of all pronumerals.

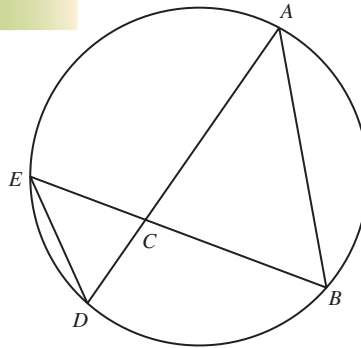


9. A circle with centre  $O$  has radius  $r$  and chord  $AB = x$ .



Show that  $CD = \frac{2r + \sqrt{4r^2 - x^2}}{2}$ .

- 10.

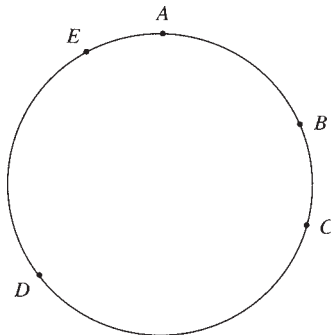


- (a) Prove that triangles  $ABC$  and  $CDE$  are similar.  
 (b) Show that  $AC \cdot CD = BC \cdot CE$ .

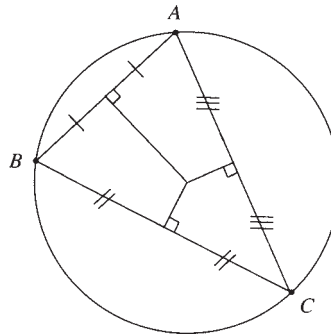
## Concyclic Points

EXTENSION

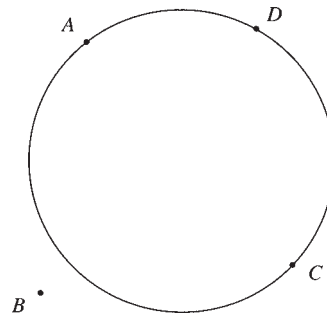
Concyclic points are points that lie on the circumference of a circle.



Any 3 **non-collinear** points are **concyclic**. They lie on a unique circle, with centre at the point of intersection of the perpendicular bisectors of the intervals joining these points.

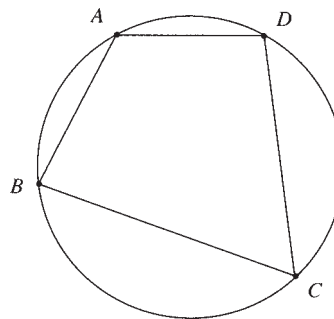


Four or more non-collinear points **may not** necessarily lie on a circle.

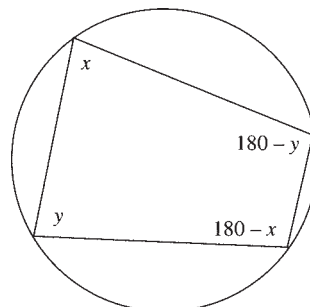


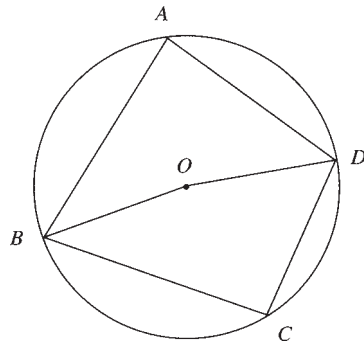
### Cyclic quadrilaterals

A **cyclic quadrilateral** is a figure whose **4 vertices** are **concyclic points**.



The opposite angles in a cyclic quadrilateral are supplementary.



**Proof**

Join  $B$  and  $D$  to  $O$ .

$$\text{Obtuse } \angle DOB = 2\angle A \quad (\angle \text{ at centre is double } \angle \text{ at circumference})$$

$$\text{Reflex } \angle DOB = 2\angle C \quad (\angle \text{ of revolution})$$

$$\text{Obtuse } \angle DOB + \text{reflex } \angle DOB = 360^\circ \quad (\angle \text{ of revolution})$$

$$\therefore 2\angle A + 2\angle C = 360^\circ$$

$$\therefore \angle A + \angle C = 180^\circ$$

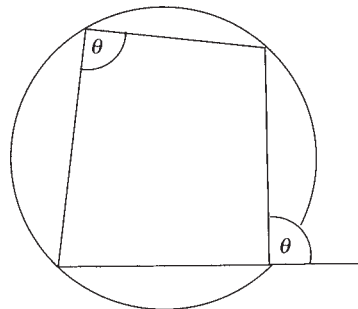
Similarly, it can be proven that  $\angle B + \angle D = 180^\circ$  by joining  $A$  and  $C$  to  $O$ .

The converse is also true:

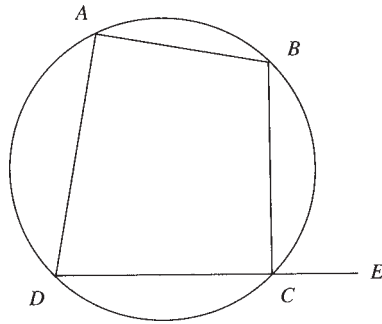
If the opposite angles of a quadrilateral are supplementary, then the quadrilateral is cyclic.

The property of opposite angles being supplementary in a cyclic quadrilateral can also be used to prove the following property:

The exterior angle at a vertex of a cyclic quadrilateral is equal to the interior opposite angle.



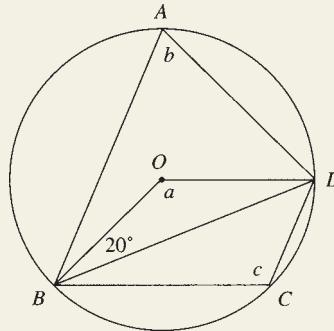


**Proof**

Let  $\angle A = x$   
 Then  $\angle BCD = 180^\circ - x$  (opposite  $\angle$ s supplementary in cyclic quadrilateral)  
 $\therefore \angle BCD + \angle BCE = 180^\circ$  ( $\angle DCE$  straight angle)  
 $\therefore \angle BCE = 180^\circ - (180^\circ - x)$   
 $\quad = 180^\circ - 180^\circ + x$   
 $\quad = x$   
 $\therefore \angle A = \angle BCE$

**EXAMPLE**

Evaluate  $a$ ,  $b$  and  $c$ .

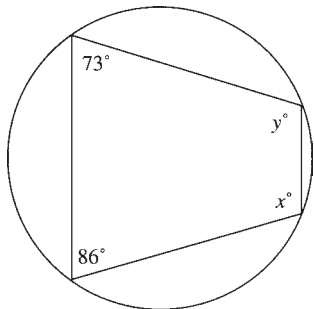
**Solution**

$OB = OD$  (equal radii)  
 $\therefore \angle ODB = 20^\circ$  (base  $\angle$ s of isosceles  $\Delta$  equal)  
 $a + 20^\circ + 20^\circ = 180^\circ$  ( $\angle$  sum of  $\Delta$ )  
 $\therefore a = 180^\circ - 40^\circ$   
 $\quad = 140^\circ$   
 $b = 70^\circ$  ( $\angle$  at centre double  $\angle$  at circumference)  
 $c = 180^\circ - 70^\circ$  (opposite  $\angle$ s in cyclic quadrilateral)  
 $\quad = 110^\circ$

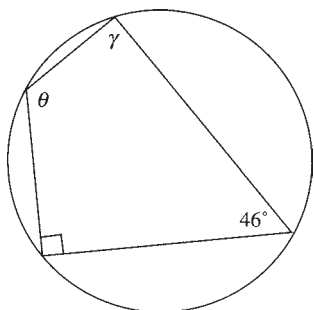
### 9.3 Exercises

1. Find the values of all pronumerals.

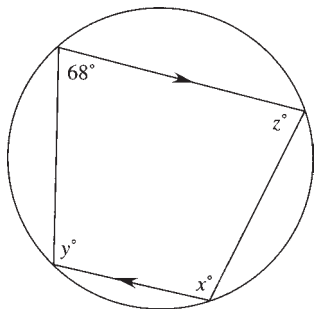
(a)



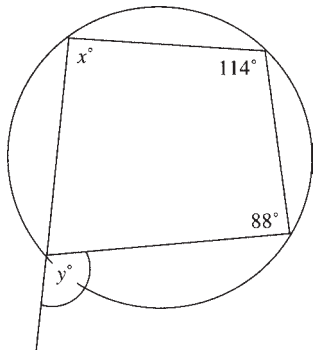
(b)



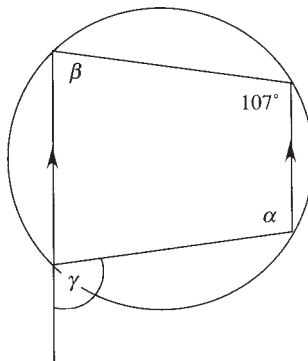
(c)



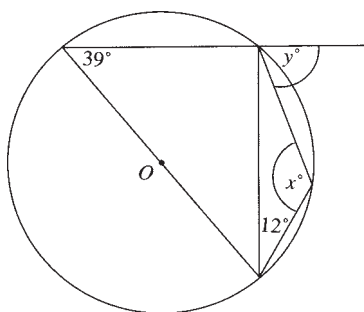
(d)



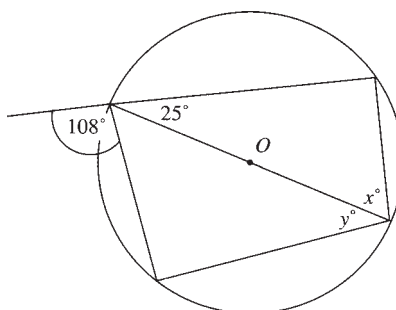
(e)



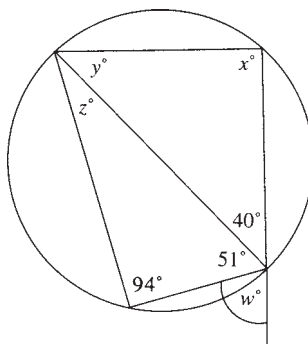
(f)



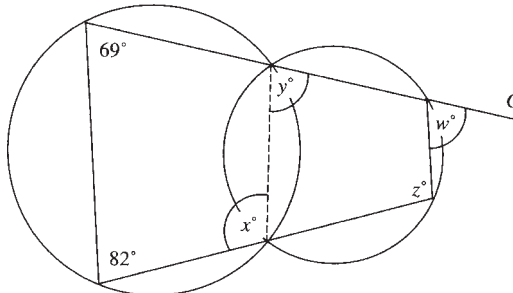
(g)

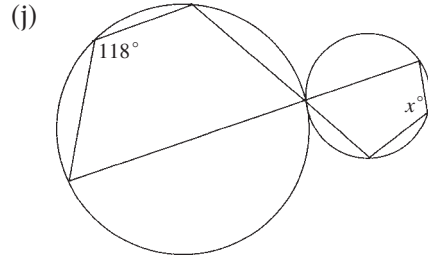


(h)

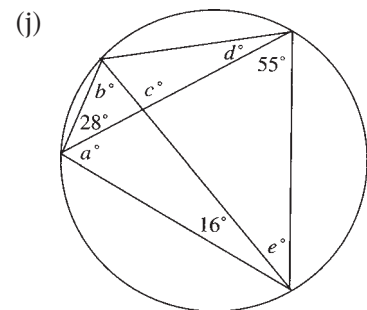
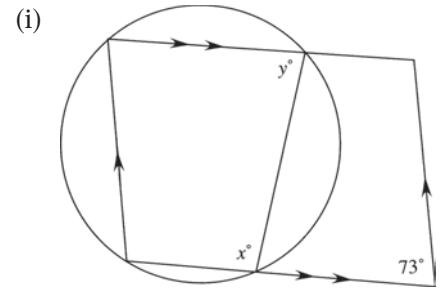
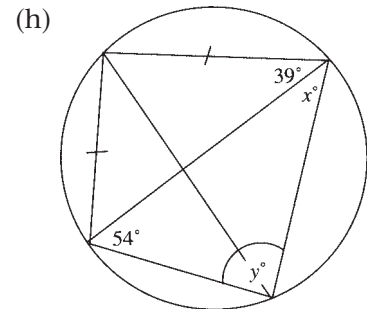
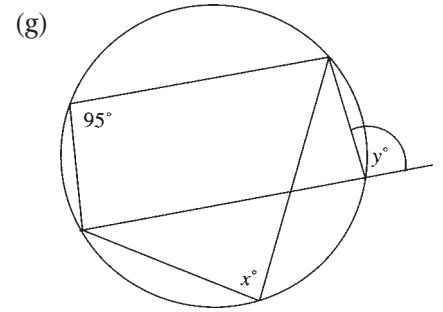
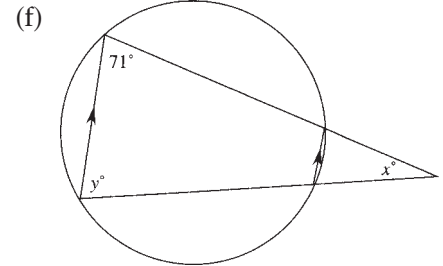
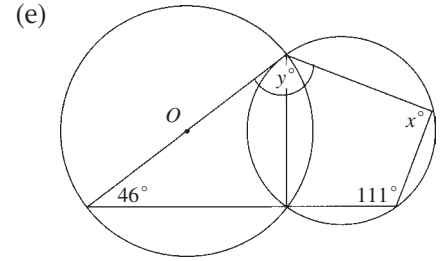
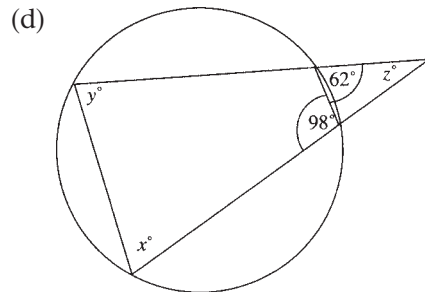
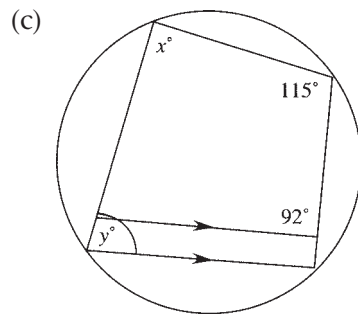
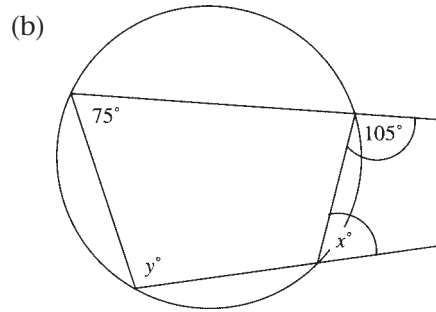
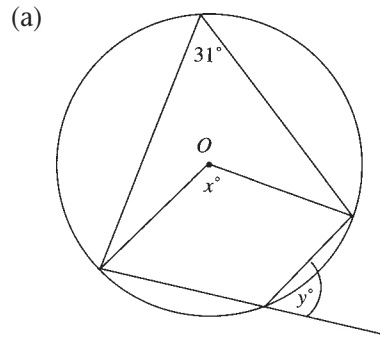


(i)

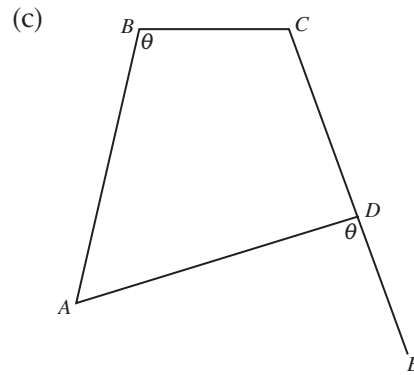
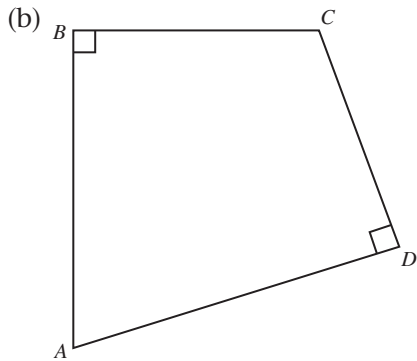
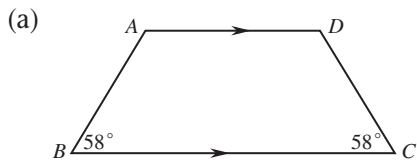




2. Find the values of all pronumerals.



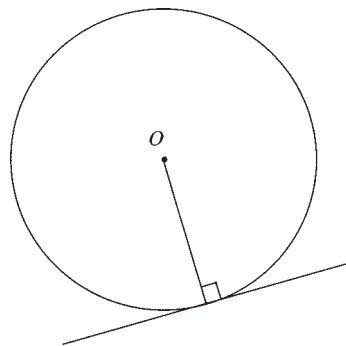
3. Show that  $ABCD$  is a cyclic quadrilateral.



## Tangent Properties

EXTENSION

The tangent to a circle is perpendicular to the radius drawn from the point of contact.



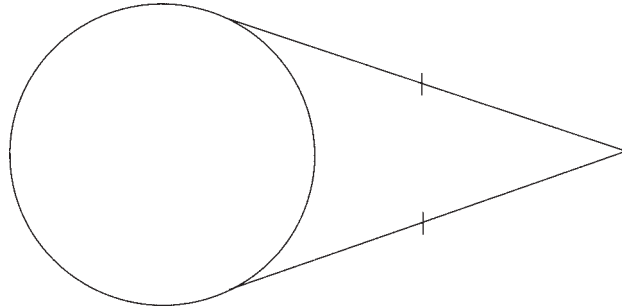
*The perpendicular distance is the shortest distance — any other distance would be greater than the radius.*

The converse is also true:

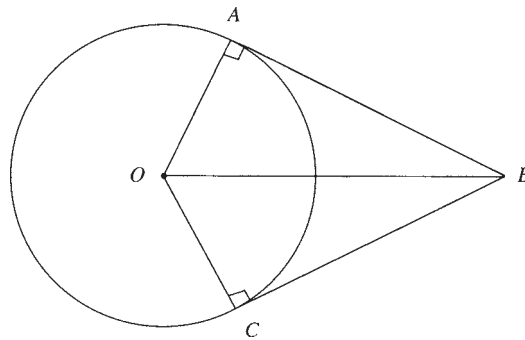
The line perpendicular to the radius at the point where it meets the circle is a tangent to the circle at that point.

Here is another property of tangents to a circle:

Tangents to a circle from an exterior point are equal.



**Proof**



Join  $OB$

$$\angle A = \angle C = 90^\circ \quad (\text{tangent} \perp \text{radius})$$

$OB$  is common

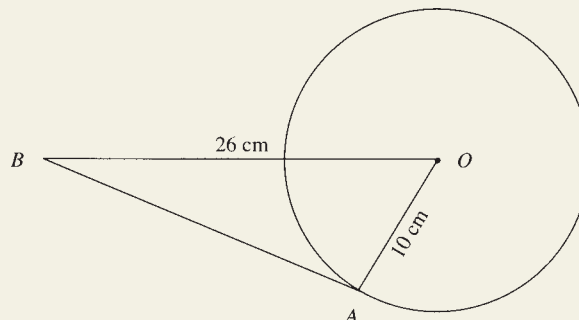
$$OA = OC \quad (\text{equal radii})$$

$\therefore$  by RHS,  $\triangle OAB \equiv \triangle OCB$

$\therefore AB = CB \quad (\text{corresponding sides in congruent } \triangle\text{s})$

### EXAMPLE

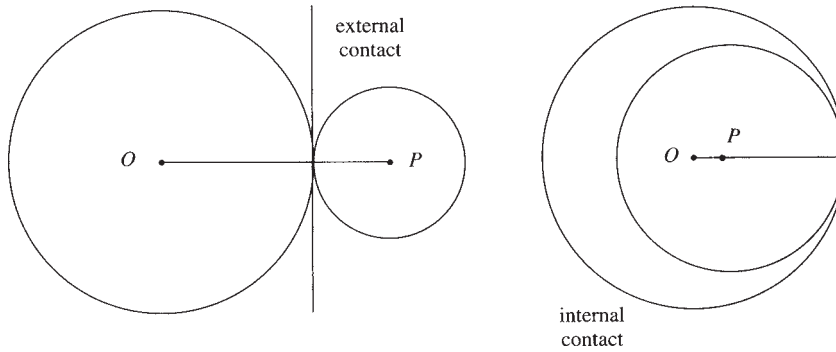
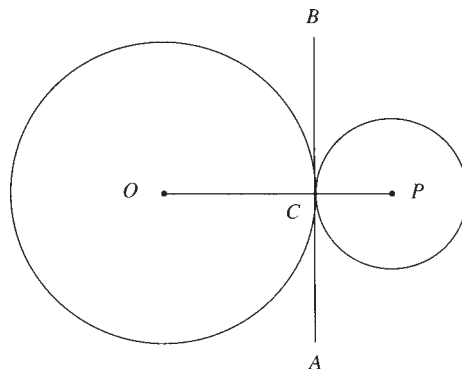
A circle with centre  $O$  and radius 10 cm has a tangent  $AB$  drawn to it where  $OB = 26$  cm. Find the length of  $AB$ .



**Solution**

$$\begin{aligned}
 OA &= 10 && \text{(radius)} \\
 \angle OAB &= 90^\circ && \text{(tangent} \perp \text{radius)} \\
 \therefore OB^2 &= OA^2 + AB^2 \\
 26^2 &= 10^2 + AB^2 \\
 676 &= 100 + AB^2 \\
 576 &= AB^2 \\
 \therefore AB &= \sqrt{576} \\
 &= 24 \text{ cm}
 \end{aligned}$$

When two circles touch, the line through their centres passes through their point of contact.

**Proof**

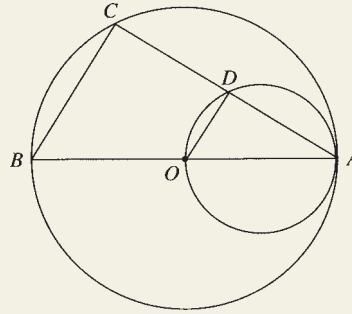
$AB$  is a tangent to circle with centre  $O$   
 $\therefore \angle OCB = 90^\circ$  (tangent  $\perp$  radius)  
 $AB$  is a tangent to circle with centre  $P$   
 $\therefore \angle PCB = 90^\circ$  (similarly)  
 $\angle OCB + \angle PCB = 90^\circ + 90^\circ$   
 $= 180^\circ$   
 $\therefore OCP$  is a straight line.

*You could also prove this result for when circles touch internally.*

## EXAMPLE

Two circles touch at  $A$  and the larger circle has centre  $O$ . Prove

- $\triangle ABC$  and  $\triangle AOD$  are similar
- $CB \parallel DO$
- $BC = 2DO$ .



## Solution

- $OA$  is a diameter of small circle (line through centres passes through point of contact)

$\therefore \angle ODA = 90^\circ$  ( $\angle$  in semicircle)

Since  $AB$  is a diameter of the larger circle,

$\angle BCA = 90^\circ$  (similarly)

$\therefore \angle BCA = \angle ODA$

$\angle A$  is common

$\therefore \triangle ABC \parallel \triangle AOD$
- $\angle BCA = \angle ODA$  [from (a)]

These are equal corresponding angles.

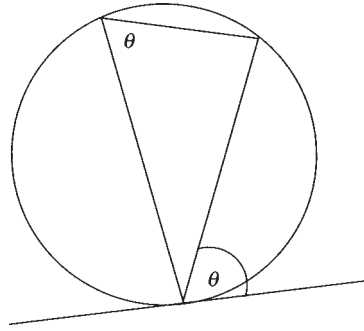
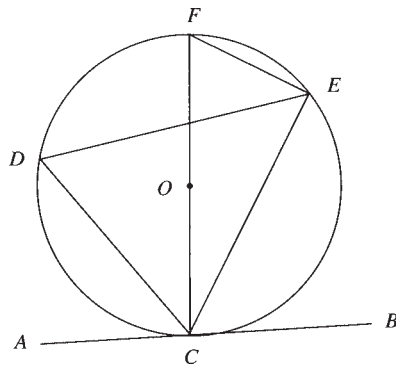
$\therefore CB \parallel DO$
- $AB = 2OA$  ( $OA$  radius)

$\therefore \frac{AB}{OA} = 2$

$\therefore \frac{AB}{OA} = \frac{AC}{AD} = \frac{BC}{DO} = 2$  (by similar  $\Delta$ s)

$\therefore BC = 2DO$

The angle between a tangent and a chord through the point of contact is equal to the angle in the alternate segment.

**Proof**

Draw in diameter  $CF$  and join  $EF$ .

Let  $\angle ECB = x$ .

$$\angle FCB = 90^\circ \quad (\text{tangent} \perp \text{radius})$$

$$\therefore \angle FCE = 90^\circ - x$$

$$\angle FEC = 90^\circ \quad (\angle \text{ in semicircle})$$

$$\therefore \angle EFC = 180^\circ - (90^\circ + 90^\circ - x) \quad (\text{angle sum of } \Delta)$$

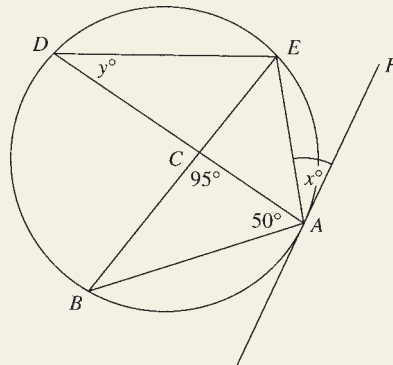
$$= x$$

$$\angle EFC = \angle EDC \quad (\text{angles in same segment})$$

$$\therefore \angle EDC = \angle ECB$$

**EXAMPLE**

Evaluate  $x$  and  $y$ .





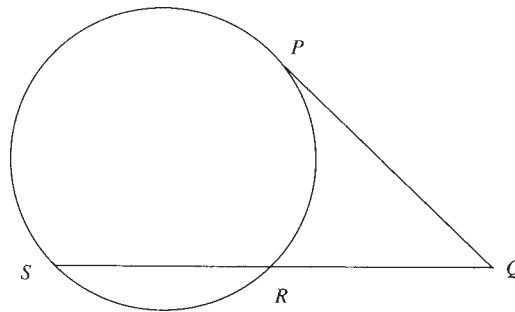
**Solution**

$$\begin{aligned}\angle ABC &= 180^\circ - (95^\circ + 50^\circ) && (\angle \text{sum of } \Delta) \\ &= 35^\circ\end{aligned}$$

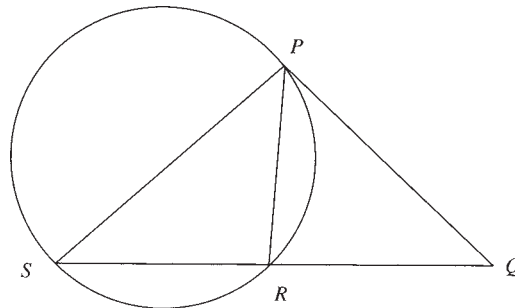
$$\therefore x = 35^\circ \quad (\angle \text{ s in alternate segment})$$

$$y = 35^\circ \quad (y \text{ and } \angle ABC \text{ in same segment})$$

The square of the length of the tangent from an external point is equal to the product of the intercepts of the secant passing through this point.



$PQ^2 = QR \cdot QS$  where  $PQ$  is a tangent to the circle.

**Proof**

$$\angle QPR = \angle PSR$$

(angles in alternate segments)

$$\angle Q \text{ is common}$$

$$\therefore PQR \sim \Delta SPQ$$

$$\therefore \frac{PQ}{QS} = \frac{PR}{SP} = \frac{QR}{PQ}$$

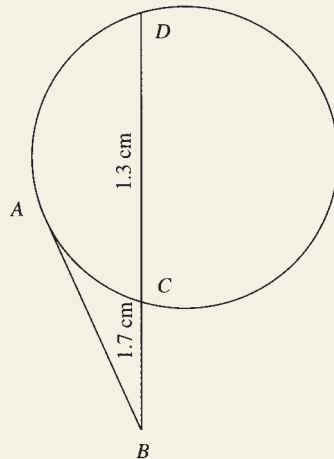
$$\therefore \frac{PQ}{QS} = \frac{QR}{PQ}$$

$$PQ^2 = QR \cdot QS$$

The third pair of angles is equal by angle sum of a triangle.

## EXAMPLE

$AB$  is a tangent to the circle and  $CD = 1.3$  cm,  $BC = 1.7$  cm. Find the length of  $AB$ , correct to 1 decimal place.

**Solution**

$$\begin{aligned} BD &= 1.3 + 1.7 \\ &= 3 \end{aligned}$$

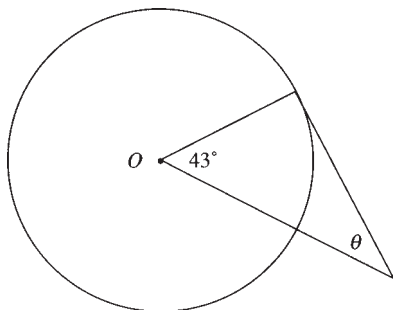
$$\begin{aligned} AB^2 &= BC \cdot BD \\ &= 1.7 \times 3 \\ &= 5.1 \end{aligned}$$

$$\begin{aligned} AB &= \sqrt{5.1} \\ &= 2.3 \text{ cm correct to 1 decimal place.} \end{aligned}$$

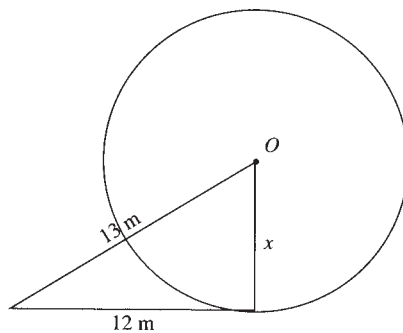
## 9.4 Exercises

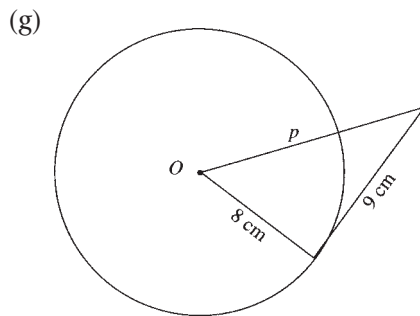
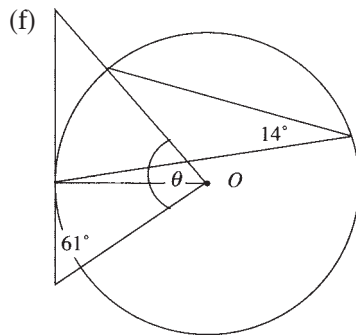
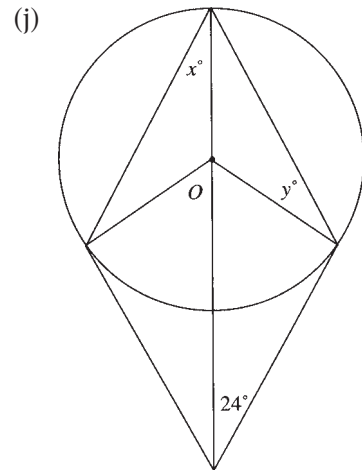
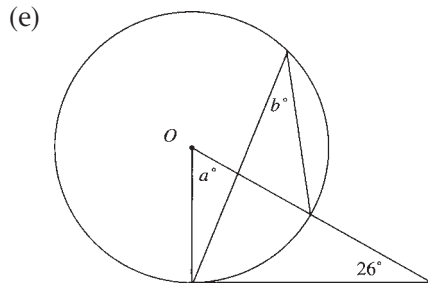
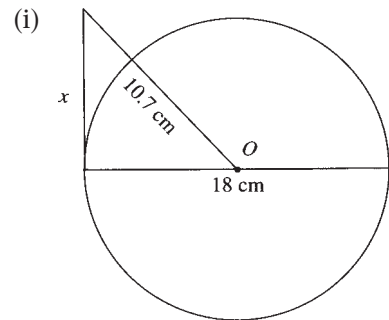
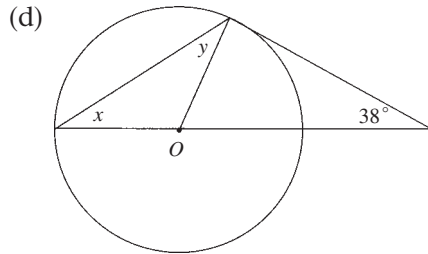
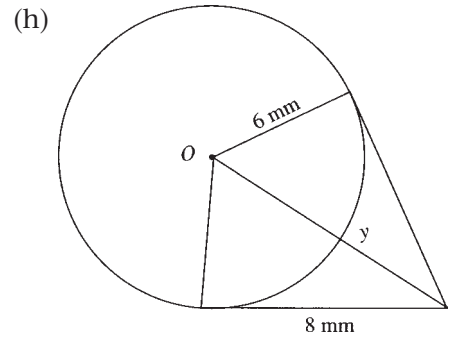
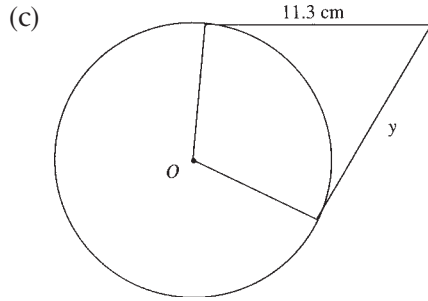
1. Find the values of all pronumerals.

(a)

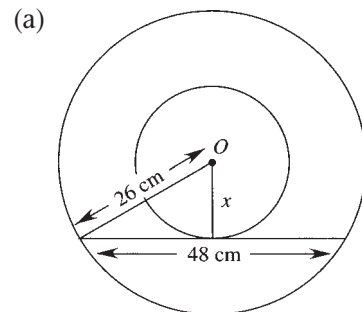


(b)

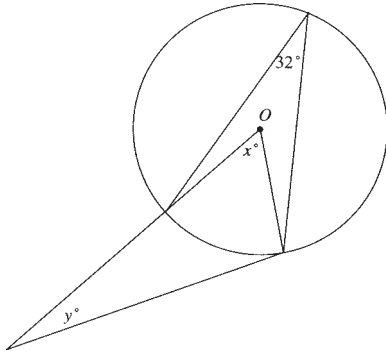




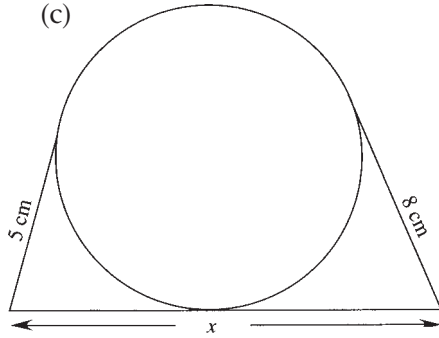
2. Find the values of all pronumerals (all external lines are tangents to the circles).



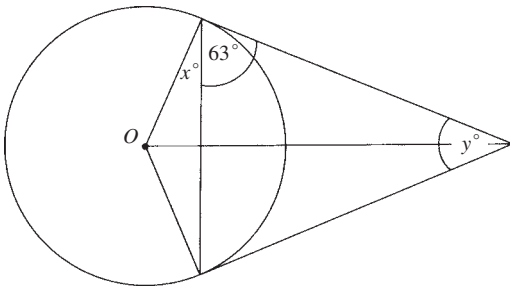
(b)



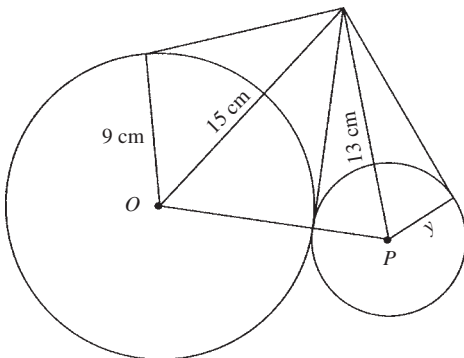
(c)



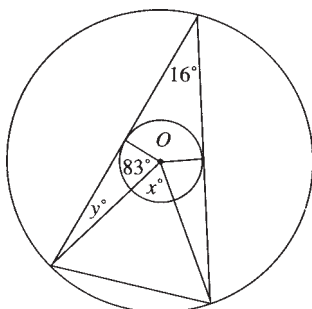
(d)



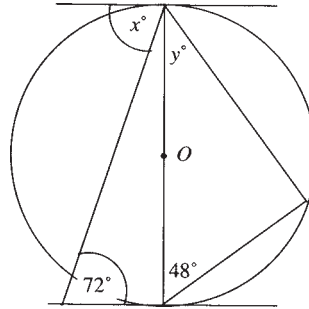
(e)



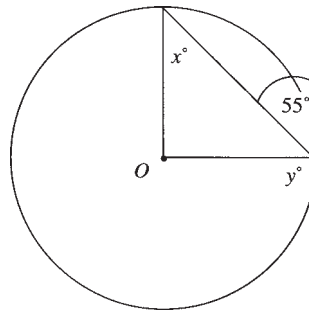
(f)



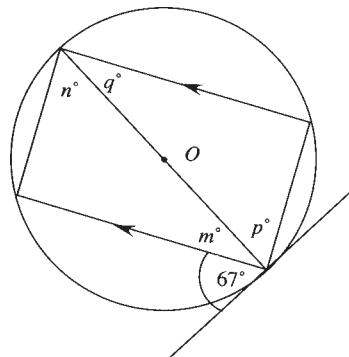
(g)



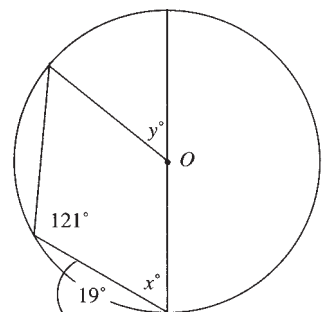
(h)



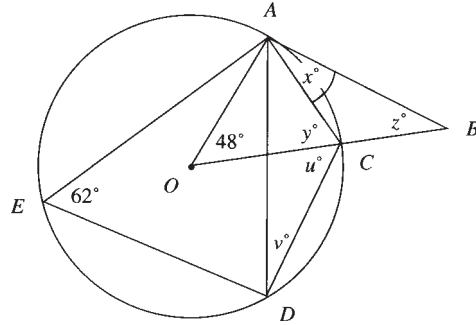
(i)



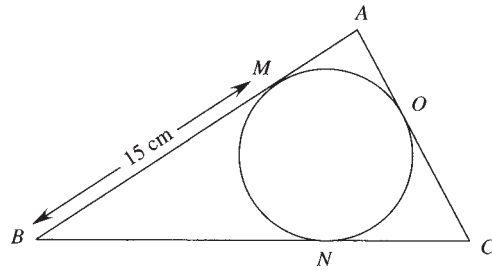
(j)



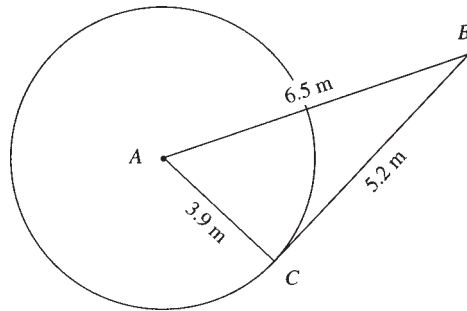
3. Find the values of all pronumerals, giving reasons for each step of your working ( $O$  is the centre of circle,  $AB$  is a tangent).



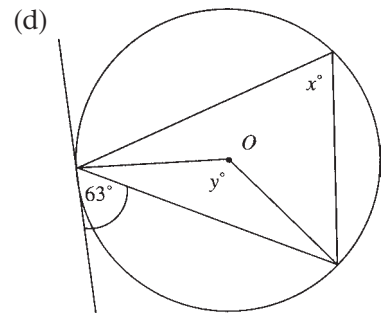
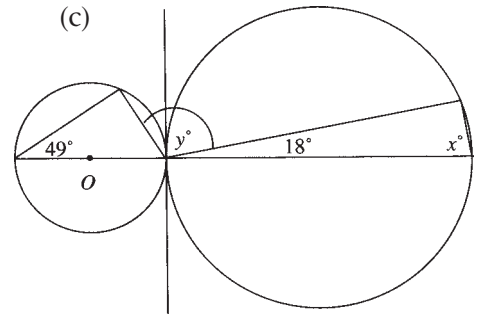
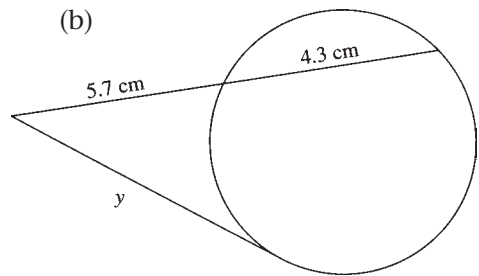
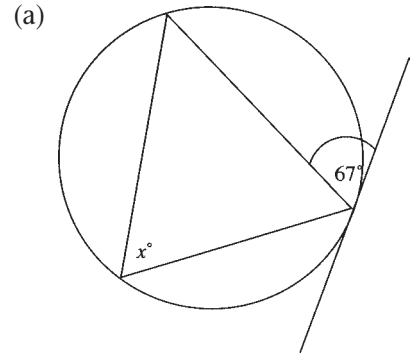
4.  $AB$ ,  $BC$  and  $AC$  are tangents, with  $AB = 24$  cm,  $BC = 27$  cm and  $BM = 15$  cm. Find the length of  $AC$ .

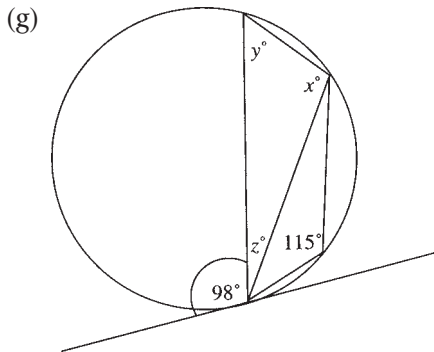
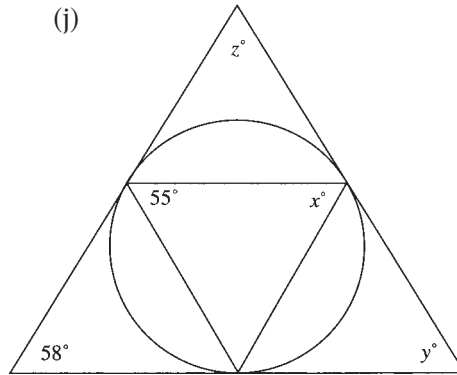
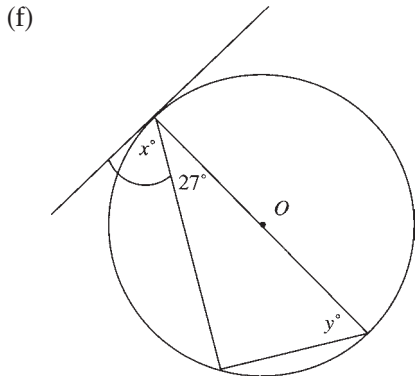
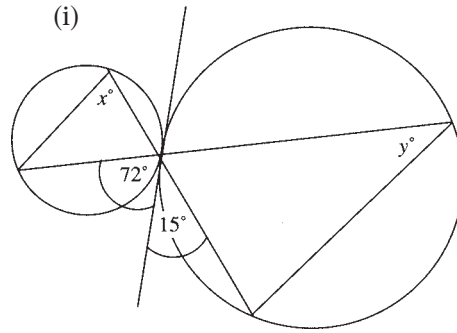
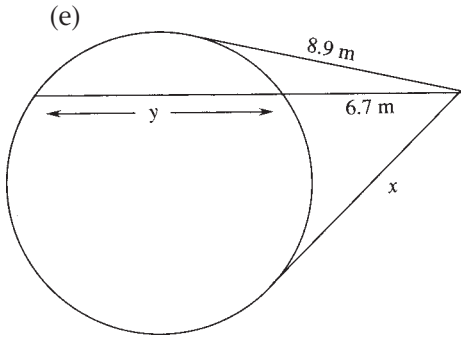


5.  $AB = 6.5$  m,  $AC = 3.9$  m and  $BC = 5.2$  m. Prove  $A$  lies on a diameter of the circle, given  $BC$  is a tangent to the circle.

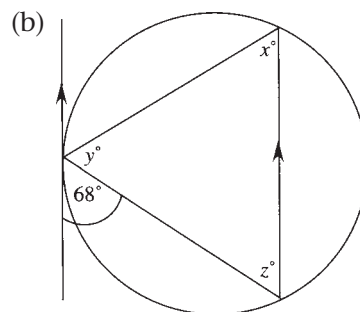
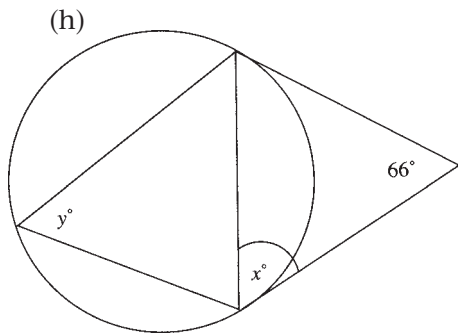
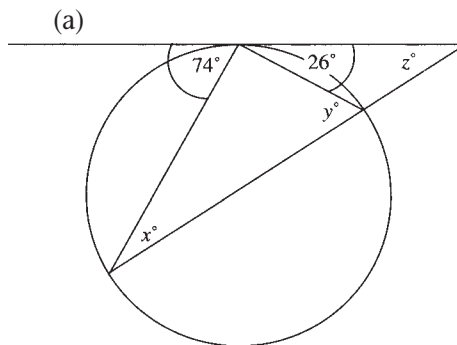


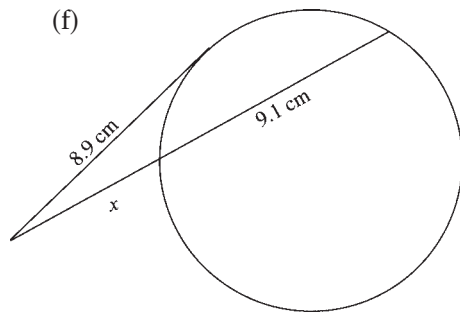
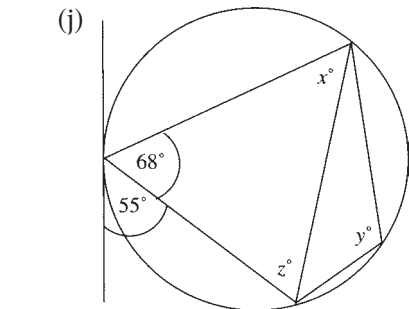
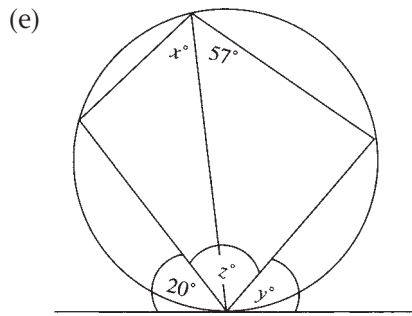
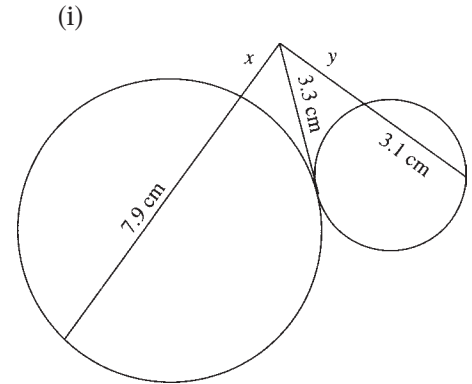
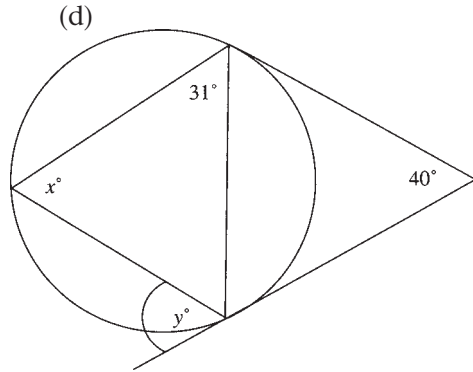
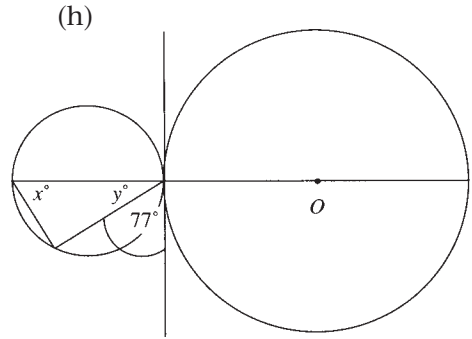
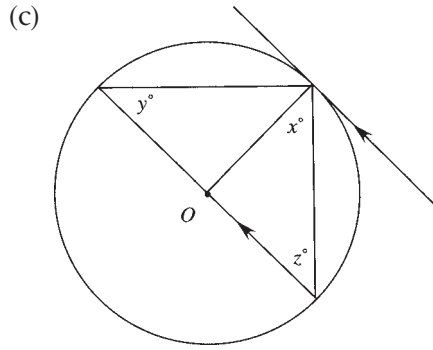
6. Find the values of all pronumerals ( $O$  is the centre of each circle; all external lines are tangents).



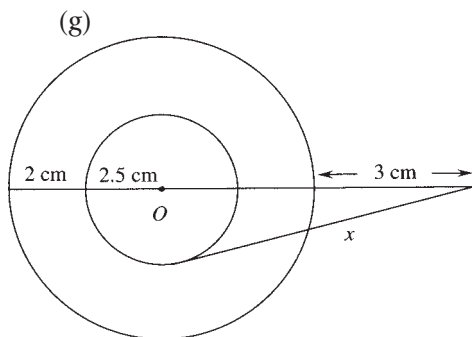
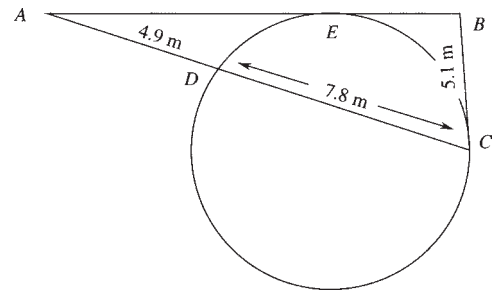


7. Find the values of all pronumerals.



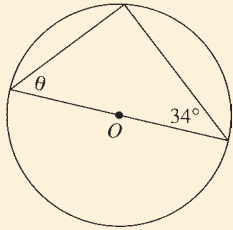


8. Find  $AB$ , given  $AD = 4.9$  m,  $BC = 5.1$  m and  $CD = 7.8$  m.

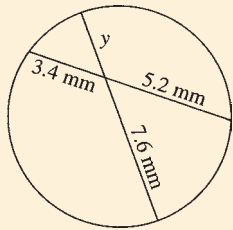


## Test Yourself 9

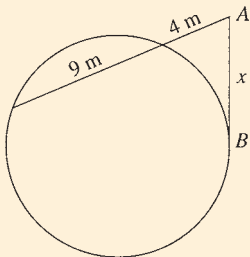
1.  $O$  is the centre of the circle. Evaluate  $\theta$ .



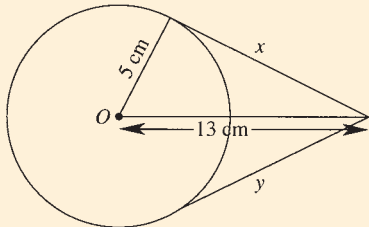
2. Evaluate  $y$  to 1 decimal place.



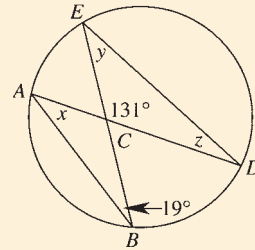
3.  $AB$  is a tangent to the circle. Find the value of  $x$  to 1 decimal place.



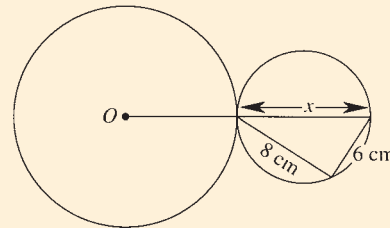
4.  $O$  is the centre of the circle. Find the length of tangents  $x$  and  $y$ .



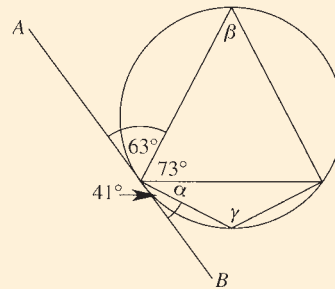
5. Evaluate  $x$ ,  $y$  and  $z$ , giving reasons for each step of your working.



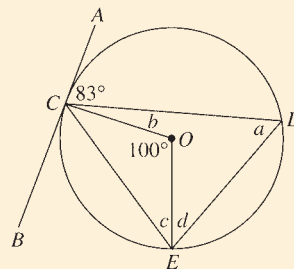
6.  $O$  is the centre of the larger circle. Find the value of  $x$ .



7.  $AB$  is a tangent to the circle. Evaluate  $\alpha$ ,  $\beta$  and  $\gamma$ .

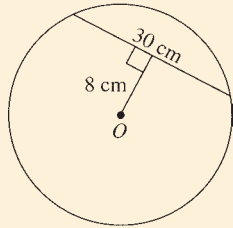


8.  $O$  is the centre of the circle, and  $AB$  is a tangent. Evaluate  $a$ ,  $b$ ,  $c$  and  $d$ , giving reasons for each step of your working.

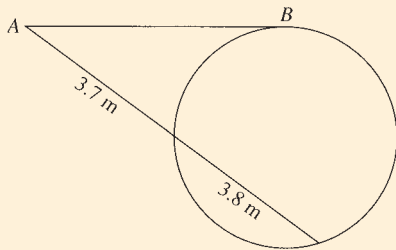




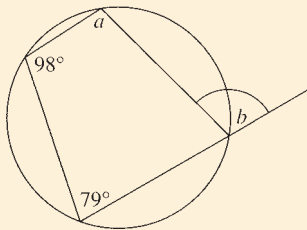
9. Find the length of the radius of the circle.  $O$  is the centre.



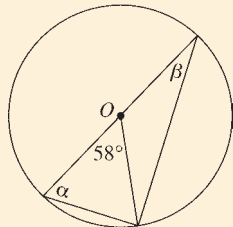
10. Find the length of tangent  $AB$ .



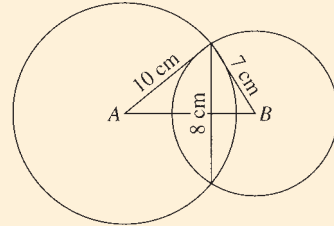
11. Evaluate  $a$  and  $b$ .



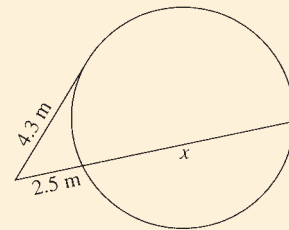
12.  $O$  is the centre of the circle. Find the value of  $\alpha$  and  $\beta$ .



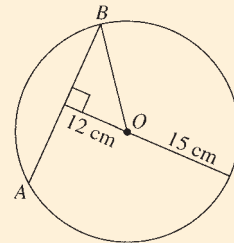
13. Calculate the length of  $AB$  to 3 significant figures, given that  $A$  and  $B$  are the centres of the circles.



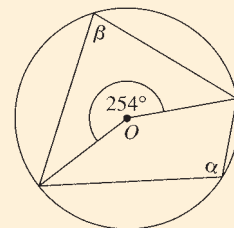
14. Find the value of  $x$  to 1 decimal place.



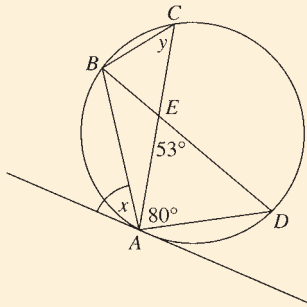
15. Find the length of  $AB$ .



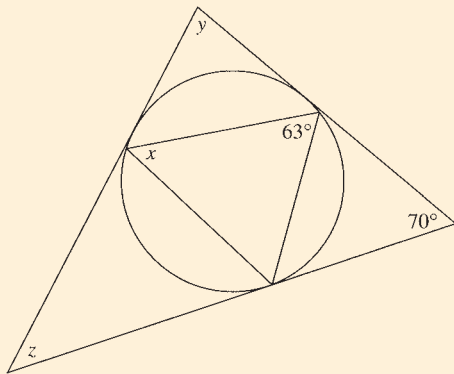
16. Evaluate  $\alpha$  and  $\beta$ .



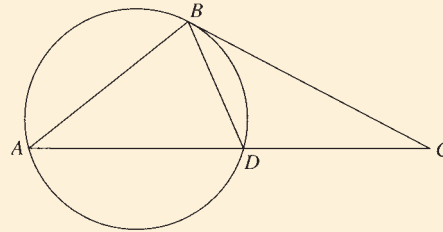
17. Evaluate  $x$  and  $y$ , giving reasons for your working.



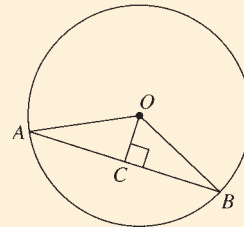
18. Evaluate  $x$ ,  $y$  and  $z$ .



19. Prove that  $\triangle BCD$  is similar to  $\triangle ABC$ .

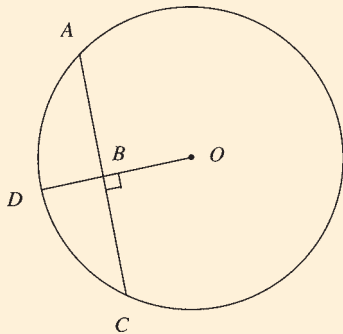


20.  $O$  is the centre of the circle.  
 (a) Prove that  $\triangle OAC$  and  $\triangle OBC$  are congruent.  
 (b) Show that  $OC$  bisects  $AB$ .

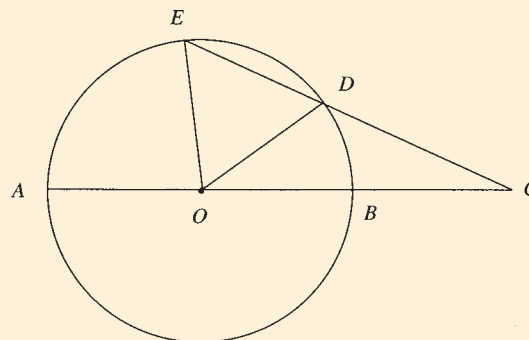


## Challenge Exercise 9

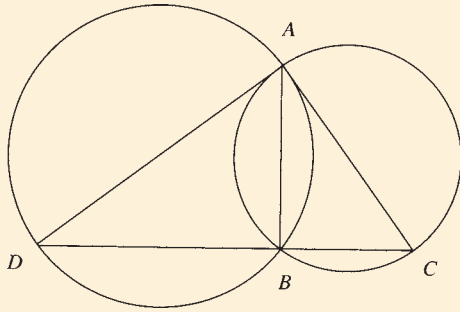
1. Find the length of the radius, to the nearest centimetre, if  $AC = 10$  cm and  $BD = 3$  cm.



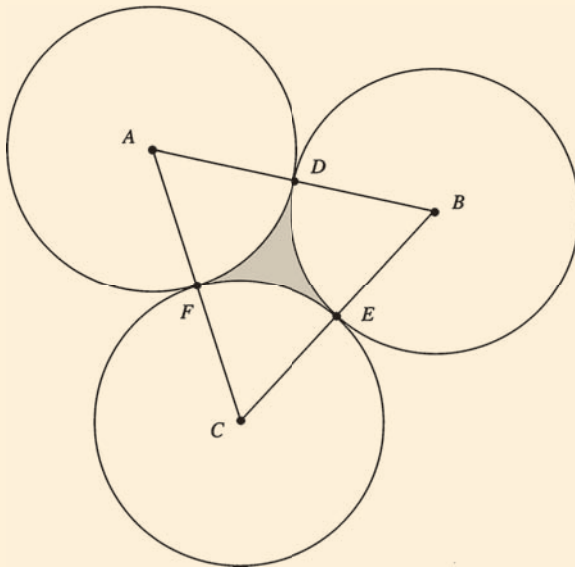
2. In the circle below with centre  $O$ ,  $OD = DC$ . Prove  $\angle AOE = 3\angle DCB$ .



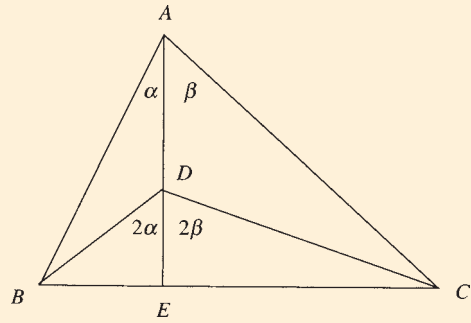
3. Two circles meet at points  $A$  and  $B$ . A tangent to each circle is drawn from  $A$  to meet the circles at  $D$  and  $C$ . Prove  $\angle DAC = 90^\circ$ .



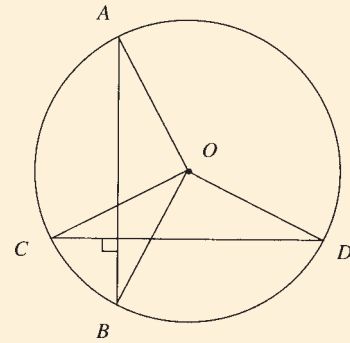
4. Three equal circles touch each other, as in the figure.  
 (a) Prove that the triangle with vertices the centres of the circles is equilateral.  
 (b) Find the perimeter of the curved figure  $DEF$  in terms of the radius  $r$  of the circles.  
 (c) Find the exact area of the shaded region.



5. The triangles below have  $\angle BDE = 2\angle BAD$  and  $\angle CDE = 2\angle CAD$ . Prove that a circle can be drawn through  $A$ ,  $B$  and  $C$  with centre  $D$ .

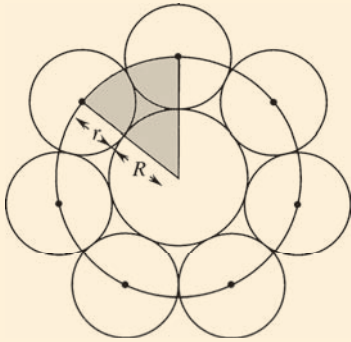


6. Two chords  $AB$  and  $CD$  intersect at  $90^\circ$ . Prove, for obtuse  $\angle AOD$ ,  $\angle AOD + \angle COB = 180^\circ$  where  $O$  is the centre of the circle.



7. Prove that any kite  $ABCD$  with  $\angle ADC = \angle ABC = 90^\circ$  is a cyclic quadrilateral with diameter  $AC$ .

8. A large circle with radius  $R$  is surrounded by 7 smaller circles with radius  $r$ . A circle is drawn through the centres of the smaller circles. If  $R = \frac{3r}{2}$ , find the shaded area in terms of  $r$ .



9. Prove that if an interval subtends equal angles at two points on the same side of it, then the endpoints of the interval and the two points are concyclic.
10. Prove that if both pairs of opposite angles in a quadrilateral are supplementary, then the quadrilateral is cyclic.